Higher Extensions of Lie Algebroids and Application to Courant Algebroids*

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Abstract

A Lie algebra integrates to a Lie group. In this paper, we find a "group-like" integration object for an exact Courant algebroid. The idea is that we first view an exact Courant algebroid as an extension of the tangent bundle by its coadjoint representation (up to homotopy), then we perform the integration by the usual method of integration of an extension.

Contents

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1 Introduction

Recently, many efforts have been made to integrate Courant algebroids, that is, to find a global object[1] associated to a Courant algebroid. Perhaps one reason for the interest in such objects is that the standard Courant algebroid serves as the generalized tangent bundle of a generalized complex manifold in the sense of Hitchin and Gualtieri [Hit03, Gua]. Thus the integration will help to understand the global symmetry of such manifolds.

Courant algebroids (see Section 2.3) were first introduced in [LWX97] to study doubles of Lie bialgebroids. An equivalent definition via graded manifolds was given by Roytenberg in [Roy]. Then [RW98] discovered the relationship between Courant algebroids and L_∞-algebras, which was an indication of the higher structures behind Courant algebroids. Here we briefly recall that an exact Courant algebroid $TM \oplus T^*M$ with Ševera class $[H]$, where $H \in \Omega^3(M)$, has antisymmetric bracket

$$[\![X+\xi, Y+\eta]\!] \triangleq [X,Y] + L_X\eta - L_Y\xi + \frac{1}{2}d(\xi(Y) - \eta(X)) + i_{X\wedge Y}H. \tag{1}$$

When $H = 0$, this defines a standard Courant algebroid. However, the bracket $[\![\cdot,\cdot]\!]$ does not satisfy the Jacobi identity, and this is the major obstruction to finding "group-like objects which integrate these not-quite-Lie algebras" (as stated in [KW01], which is perhaps the first paper to explore the integration of Courant brackets).

Mehta and Tang [MT] apply the Artin-Mazur construction to Mackenzie's symplectic double groupoid and obtain a symplectic Lie 2-groupoid corresponding to the standard Courant algebroid (namely the one with trivial Ševera class); Li-Bland and Ševera [LBŠ], on the other hand, start with a local symplectic Lie 2-groupoid, and rigorously verify via Ševera's differentiation of higher Lie groupoids [Ševa] that this local symplectic Lie 2-groupoid differentiates to an exact Courant algebroid (possibly with a non-trivial Ševera class). In fact, [MT] focuses more on the authors' procedure to pass from double groupoids to 2-groupoids and contains examples other than the integration of the standard Courant algebroid. The Lie 2-groupoids in these two papers are fundamentally the same. However, the symplectic forms are different. Thus the uniqueness of symplectic forms is in a certain sense, an open question.

In this paper, we use another method to do integration: we realize an exact Courant algebroid as an extension of the Lie algebroid TM by its representation up to homotopy[2] $T^*M \xrightarrow{\mathrm{Id}} T^*M$ with an extension class $[(c_2, c_3)] \in H^2(TM, T^*M \xrightarrow{\mathrm{Id}} T^*M)$ expressed by the Ševera class,

$$0 \longrightarrow \big(T^*M \xrightarrow{\mathrm{Id}} T^*M\big) \longrightarrow \text{Courant algebroid} \longrightarrow TM \longrightarrow 0.$$

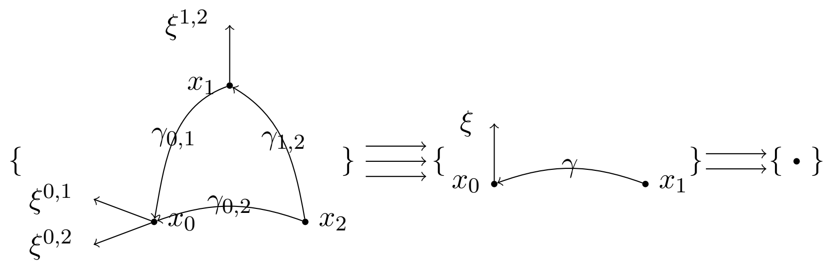
[1] For example, a global object corresponds to a Lie algebra is a Lie group.

[2] The concept of representation up to homotopy of a Lie algebroid is an extension of Lada-Markl's L_∞-modules to the context of Lie algebroids [LM95]. It has been developed recently by Abad and Crainic [ACb].

However since $H^2(TM, T^*M \xrightarrow{\mathrm{Id}} T^*M) = 0$, the extension is always trivial regardless of the Ševera class. Thus an exact Courant algebroid is always isomorphic to the standard Courant algebroid as an NQ manifold even if the Ševera class is non-zero. This isomorphism is also observed in [LBŠ, Prop. 2]; however, as observed there, it does not preserve the symplectic structure.

Then we perform the integration of the extension and obtain a Lie 2-groupoid, which is again always (regardless of the Ševera class) isomorphic to the following Lie 2-groupoid

$$\Pi_1(M)^{\times 2} \times_{M^{\times 3}} (T^*M)^{\times 3} \Rrightarrow \Pi_1(M) \times_M T^*M \rightrightarrows M, \tag{2}$$



where $\Pi_1(M) = \tilde{M} \times \tilde{M}/\pi_1(M)$ is the fundamental groupoid of M with $\tilde{M}$ the simply connected cover of M. This Lie 2-groupoid is the same (locally) as in [MT, LBŠ]. Thus, in short, all of the three papers give the same integration. However we emphasis here that there are several equivalent viewpoints towards 2-groupoids (see Section 5.2). While the other two papers take the viewpoint of Kan complexes, we take the viewpoint of categorification of groupoid à la Baez, for our convenience.

We must mention that in this paper we do not perform rigorous differentiation partially because this is the merit of [LBŠ] (only in Remark 5.5 we sketch a possible differentiation method), and we do not deal with the symplectic form. Integrating the symplectic form systematically involves integrating morphisms of higher NQ manifolds, which we do not deal here. Thus, for the purpose of integrating Courant algebroids, this paper can be viewed more as an explanation of the origin of the Lie 2-groupoid in [LBŠ]. Moreover, [GSM10] implies a possible direct link between our paper and [MT].

On the other hand, there are also some valuable byproducts achieved in this paper: We classify the 2-term abelian extensions of a Lie algebroid A by $H^2(A, \mathcal{E})$ where $\mathcal{E}$ is a 2-term representation up to homotopy of A (Thm. 4.7). Moreover, we hope that this viewpoint via representations up to homotopy can be helpful in the case of a more general Courant algebroid coming from a Lie bialgebroid. Finally, we also bring the concept of a split Lie n-algebroid onto the surface (Def. 2.2). Courant algebroids can be described using differential geometry language as in [LWX97]. However, the method of NQ manifolds perhaps reflects more the nature of Courant algebroids, though it often involves calculations in local coordinates. The language of split Lie n-algebroids provides a way to study NQ manifolds within the differential geometry framework. Thus we hope it can be a useful tool for differential geometers in general to study problems related to NQ manifolds.

In fact, this paper is the third of a series of papers with the aim of integrating Courant algebroids via representations up to homotopy. The basic observation is to view the Courant bracket as an extension of a Lie bracket, not however via a usual representation, but via

a representation up to homotopy. In [SZb], we realize the standard Courant bracket (with trivial Ševera class) as a semidirect product of the Lie bracket of vector fields and its representation up to homotopy on the complex $C^\infty(M) \xrightarrow{d} \Omega^1(M)$. We also provide several finite dimensional examples of such semidirect product coming from omni-Lie algebras and string Lie 2-algebras. In [SZa], we integrate such semidirect products in the finite dimensional case. For a long time, we were lost on how to descend from our construction at the level of sections to the Courant algebroid itself, namely, how to realize the Courant bracket as the semidirect product of the Lie algebroid TM and its coadjoint representation up to homotopy $T^*M \xrightarrow{\mathrm{Id}} T^*M$. We must point out that after talking to David Li-Bland, we were convinced that there should be a similar construction on the vector bundle level; however, the formula we have in [SZb] is clearly neither $C^\infty(M)$-linear nor a derivation, so we are not able to push the formula down to the level of vector bundles.

Finally, we realized that our confusion comes from the fact that there are two different ways to view NQ manifolds. The concept of N-manifold was introduced by Ševera in [Šev05] and appeared informally even earlier in his letter to Weinstein [Ševb]. They are non-negatively graded manifolds ("N" stands for non-negative). Then an NQ-manifold is a non-negatively graded manifold $\mathcal{M}$ together with a degree 1 vector field Q satisfying $[Q, Q] = 0$. A degree 1 NQ-manifold is a *Lie algebroid*. We recall the procedure: a degree 1 non-negative graded manifold can be modeled by a vector bundle with shifted degree $A[1] \to M$. The function ring of $A[1]$ is the graded algebra

$$C(A[1]) = C^\infty(M) \oplus \Gamma(A^*) \oplus \Gamma(\wedge^2 A^*) \oplus \cdots \tag{3}$$

A degree 1 vector field Q is a degree 1 differential of this algebra. Equivalently, this means that we have a vector bundle morphism (called the *anchor* later on) $\rho_A : A \to TM$ and a Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ such that $Q = d_A$, where $d_A : \Gamma(\wedge^n A^*) \to \Gamma(\wedge^{n+1} A^*)$ is defined as the generalized de Rham differential

$$d_A(\xi)(X_0, \dots, X_n) = \sum_{i<j} (-1)^{i+j} \xi([X_i, X_j]_A, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots) + \sum_{i=0}^{n} (-1)^i \rho_A(X_i)(\xi(\dots, \hat{X}_i, \dots)). \tag{4}$$

The equation $[Q, Q] = 0$ is then equivalent to the condition we require for $(A, \rho_A, [\cdot,\cdot]_A)$ to be a Lie algebroid, that is, $[[X, Y]_A, Z]_A + c.p. = 0$ and $[X, fY]_A = f[X, Y]_A + \rho_A(X)(f)Y$ for any $X, Y, Z \in \Gamma(A)$ and $f \in C^\infty(M)$. However, there is another method to recover the Lie bracket on A: $\Gamma(A)$ can be viewed as the space of degree -1 vector fields on $A[1]$. Then a degree 1 homological vector field Q on $A[1]$ gives us a derived bracket $[X, Y]_A := [[Q, X], Y]$, which is exactly the Lie algebroid bracket on A corresponding to Q given above.

However, when we try to do the same for Lie n-algebroid for $n \geq 2$, we encounter a different story. First of all, to model a degree n non-negative graded manifold on a graded vector bundle requires an unnatural choice of connection (even though it is always possible). It is comparable to the fact that to single out a composition of 1-cells of a Lie n-groupoid $X_\bullet$ modeled using a Kan simplicial manifold requires an unnatural choice of a section from the horn space $X_1 \times_{X_0} X_1$ to X_2 (in this case it is not always possible[3]). However, there are still some circumstances in which a graded vector bundle arise naturally (namely a preferred connection is chosen somehow) to hold the structure of an NQ manifold. For example, a

[3]However it is always possible locally which explains the existence of the choice in the infinitesimal case. (Private conversation with Dimitry Roytenberg.)

representation up to homotopy $\mathcal{E}_n$ of a Lie algebroid A naturally give rises to such a graded vector bundle and there should be an NQ manifold structure on the semidirect product $A \ltimes \mathcal{E}_n$ (see Lemma 3.1). For this reason we still consider that our degree n N-manifold comes from a graded vector bundle $\mathcal{A} = A_0 \oplus A_{-1} \oplus \cdots \oplus A_{-n+1}$, then, similarly to (3), the function ring is the graded commutative algebra,

$$\begin{aligned} C(\mathcal{A}[1]) = & C^\infty(M) \oplus \Big[\Gamma(A_0^*)\Big] \oplus \Big[\Gamma(\wedge^2 A_{-1}^*) \oplus \Gamma(A_{-1}^*)\Big] \\ & \oplus \Big[\Gamma(\wedge^3 A_0^*) \oplus \Gamma(A_0^*) \otimes \Gamma(A_{-1}^*) \oplus \Gamma(A_{-2}^*)\Big] \oplus \cdots . \end{aligned} \tag{5}$$

where A_{-i}^* has degree $i+1$ and $C^\infty(M)$ lies in degree 0. Then a homological degree 1 vector field Q gives us an anchor ρ and various brackets l_i for $i = 1, \ldots, n+1$ (see Def. 2.2). We call such an object a *split Lie n-algebroid*. It turns out that if we begin with a degree 2 NQ manifold–for example $T^*[2]T[1]M$, the one corresponding to an exact Courant algebroid (see Section 2.3), then the Courant bracket arises as the derived bracket; however the derived bracket is **different** from l_2 no matter how we choose the splitting. This picture, which is different from the degree-1 case, clarifies our confusion: what we should obtain from semidirect product construction is l_2, but not the Courant bracket, even though they are equivalent in a certain sense.

Notations: Throughout the paper, we use $\mathcal{E}$ to denote the 2-term complex of vector bundles $\partial : E_{-1} \longrightarrow E_0$ and $E_\bullet$ is the corresponding abelian Lie 2-groupoid (Example 5.3). $(A, [\cdot,\cdot]_A, \rho_A)$ is a Lie algebroid and d_A is the corresponding differential defined in (4). We use $\mathcal{G}$ to denote a Lie groupoid $G_1 \rightrightarrows G_0$ and $\mathcal{G}$ to denote the Lie groupoid $G_2 \rightrightarrows G_1$ in a semistrict Lie 2-groupoid $G_2 \rightrightarrows G_1 \rightrightarrows G_0$. d is the usual de Rham differential. We use Id to denote the identity map of vector bundles and id_M(*resp.* id_{G_0}) is the identity map on the manifold M(*resp.* G_0).

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2 Preliminaries

In this section we recall important concepts that we will use in our paper.

2.1 NQ manifolds and Lie n-algebroids

Recall that an NQ-manifold is a non-negatively graded manifold $\mathcal{M}$ together with a degree 1 vector field Q satisfying $[Q, Q] = 0$, i.e. a linear operator Q on $C^\infty(\mathcal{M})$ that raises the degree by one and satisfying $Q^2 = 0$ and

$$Q(fg) = Q(f)g + (-1)^{|f|} fQ(g), \quad \forall\ f, g \in C^\infty(\mathcal{M}).$$

It is well known that a degree 1 NQ manifold is a Lie algebroid, thus we are attempt to give the following definition,

Definition 2.1 *A Lie n-algebroid is an NQ-manifold of degree n.*

Some effort towards this direction is made in [Vor]. Here we want to focus specifically on split NQ manifolds since it turns out that semidirect products, and furthermore, extensions of Lie algebroids by their representations up to homotopy, provide some natural examples of these split NQ manifolds.

Recall that a split degree n N-manifold $\mathcal{A}$ is a non-negatively graded vector bundle $A_0 \oplus A_{-1} \oplus A_{-2} \oplus \cdots \oplus A_{-n+1}$ over the same base M, with the function ring in (5). Then $C(\mathcal{A})$ should be viewed as the Chevalley-Eilenberg complex of a certain higher algebroid structure on $\mathcal{A}$ with brackets l_i's and anchor $\rho : A_0 \to TM$ such that a degree 1 derivation Q can be expressed by

$$\begin{aligned} Q(f) =& \rho^*(df), \quad \forall\ f \in C^\infty(M), \\ Q(\xi_0)(x_0 \wedge y_0 + x_1) =& \rho(x_0)\langle \xi_0, y_0\rangle - \rho(y_0)\langle \xi_0, x_0\rangle - \langle \xi_0, l_2(x_0, y_0)\rangle + \langle \xi_0, l_1(x_1)\rangle, \\ Q(\xi_1)(x_0 \wedge y_0 \wedge z_0) =& \langle \xi_1, l_3(x_0, y_0, z_0)\rangle, \\ Q(\xi_1)(x_0 \wedge x_1) =& \langle \xi_1, l_2(x_0, x_1)\rangle + \rho(x_0)\langle \xi_1, x_1\rangle, \\ Q(\xi_1)(x_2) =& \langle \xi_1, l_1(x_2)\rangle, \end{aligned} \tag{6}$$

with $\xi_i \in \Gamma(A_{-i})$, and $x_i, y_i \in A_i$. As in the case of L_∞-algebras, $(C(\mathcal{A}), Q)$ being a differential graded commutative algebra, namely $Q^2 = 0$, is equivalent to all the axioms that l_i and ρ should satisfy. Finally, we summarize this equivalent viewpoint of a split NQ manifold with the following definition:

Definition 2.2 (split Lie n-algebroid) *A split Lie n-algebroid is a graded vector bundle $\mathcal{A} = A_0 \oplus A_{-1} \oplus \cdots \oplus A_{-n+1}$ over a manifold M equipped with a bundle map $\rho : A_0 \longrightarrow TM$ (called the anchor), and $n+1$ many brackets $l_i : \Gamma(\wedge^i \mathcal{A}) \longrightarrow \Gamma(\mathcal{A})$ with degree $2-i$ for $1 \le i \le n+1$, such that*

1. $$\sum_{i+j=k+1} (-1)^{i(j-1)} \sum_{\sigma} \mathrm{sgn}(\sigma) \mathrm{Ksgn}(\sigma) l_j(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(k)}) = 0, \tag{7}$$
   where the summation is taken over all $(i, k-i)$-unshuffles with $i \ge 1$ and "$\mathrm{Ksgn}(\sigma)$" is the Koszul sign for a permutation $\sigma \in S_k$, i.e.
   $$x_1 \wedge x_2 \wedge \cdots \wedge x_k = \mathrm{Ksgn}(\sigma) x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(k)}.$$

2. *l_2 satisfies the Leibniz rule with respect to ρ:*
   $$l_2(x_0, fx) = f l_2(x_0, x) + \rho(x_0)(f)x, \quad \forall\ x_0 \in \Gamma(A_0), f \in C^\infty(M), x \in \Gamma(\mathcal{A}).$$

3. *For $i \neq 2$, l_i's are $C^\infty(M)$-linear.*

Remark 2.3 *It is clear that the brackets l_i makes the space of smooth sections $\Gamma(A_0) \oplus \Gamma(A_{-1}) \oplus \cdots \oplus \Gamma(A_{-n+1})$ of a Lie n-algebroid $\mathcal{A}$ into a (infinite dimensional) Lie n-algebra.*

In this paper, we will use both viewpoints–the one using various brackets and the anchor map, and the one using NQ manifolds–to refer to a Lie n-algebroid.

Definition 2.4 *A Lie 2-algebroid morphism (isomorphism) $\mathcal{A} \to \mathcal{A}'$ is a graded vector bundle morphism f from*

$$A_0 \oplus \big[\wedge^2 A_0 \oplus A_1 \big] \oplus \big[\wedge^3 A_0 \oplus A_0 \otimes A_{-1} \oplus A_{-2} \big] \oplus \dots$$

to

$$A_0' \oplus \big[\wedge^2 A_0' \oplus A_1 \big] \oplus \big[\wedge^3 A_0' \oplus A_0' \otimes A_{-1}' \oplus A_{-2}' \big] \oplus \dots$$

such that the induced map $f^* : C(A') \to C(A)$ is a morphism (isomorphism) of NQ manifold.

Remark 2.5 *First of all, when $n = 1$ this coincides with the usual definition of Lie algebroid morphism (see e.g. [Mac05]) in terms of vector bundles, anchors and brackets, which is not as easy as one might think. Thus, we do not expect it to be easy to express a morphism of Lie n-algebroids in terms of vector bundles, anchors and brackets. However, in the case when the $\mathcal{A}$ and $\mathcal{A}'$ share the same base and the base morphism is an isomorphism, a graded vector bundle morphism f is a Lie n-algebroid morphism if and only if f preserves the anchors and f induces an L_∞-morphisms on the sections of $\mathcal{A}$ and $\mathcal{A}'$. Notice that this imply that f preserves the brackets only in an L_∞-fashion, for example when $n = 2$, we have*

$$\begin{aligned} f_0(l_2(x_0, y_0)) - l_2'(f_0(x_0), f_0(y_0)) &= l_1'(f_2(x_0, y_0)), & (8)\\ f_1(l_2(x_0, x_1)) - l_2'(f_0(x_0), f_1(x_1)) &= f_2(x_0, l_1(x_1)), & (9) \end{aligned}$$

and

$$\begin{aligned} & l_3'(f_0(x_0), f_0(y_0), f_0(z_0)) - f_1(l_3(x_0, y_0, z_0)) \\ = \; & l_2'(f_2(x_0, y_0), f_0(z_0)) + f_2(l_2(x_0, y_0), z_0) + c.p.. \qquad (10) \end{aligned}$$

However, the converse is true, that is, if f preserves the bracket strictly then f induces an L_∞-morphisms on the sections of $\mathcal{A}$ and $\mathcal{A}'$. In this case, we shall call f a **strict morphism**. *If a Lie n-algebroid morphism f induces an isomorphism on the underlying graded vector bundle of $\mathcal{A}$ and $\mathcal{A}'$, then it is an isomorphism.*

Example 2.6 It is clear that a Lie algebroid A can be viewed as a split Lie n-algebroid with $A_0 = A$, all the other $A_i = 0$, the same anchor and l_2, and all the other higher brackets equals to 0. In fact, any Lie n-algebroid $\mathcal{A}$ can be viewed as a Lie $(n+1)$-algebroid in this way. That is, $\mathcal{A}$ is a Lie $(n+1)$-algebroid with $A_{n+1} = 0$, $l_{n+2} = 0$, and all the rest kept the same.

Example 2.7 Given a complex of vector bundles $\mathcal{E}_n : E_{-(n-1)} \xrightarrow{\partial} E_{-(n-2)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} E_0$, it can be viewed as a Lie n-algebroid with $l_1 = \partial$, any remaining bracket $l_i = 0$, and the anchor $\rho = 0$. We call such a Lie n-algebroid an *abelian* Lie n-algebroid. There are similar constructions on the groupoid level as we point out in the case when $n = 2$ in Example 5.3. The integration and differentiation between these abelian higher algebroids and groupoids are given by Dold-Kan correspondence (see for example, [LBŠ, Example 7] for a detailed explanation in this direction).

2.2 Representation up to homotopy of a Lie algebroid

Consider a graded vector bundle $\mathcal{E}_n$ of degree n:

$$\mathcal{E}_n : E_{-(n-1)} \oplus E_{-(n-2)} \oplus \cdots \oplus E_0.$$

Its dual $\mathcal{E}_n^*[1]$ with degree shifted by 1 is

$$\mathcal{E}_n^*[1] : E_0^* \oplus E_{-1}^* \oplus \cdots \oplus E_{-(n-1)}^*, \quad \text{with} \deg(E_{-i}^*) = i+1.$$

Definition 2.8 [ACb] *A representation up to homotopy of a Lie algebroid A consists of a graded vector bundle $\mathcal{E}_n$ over M and an operator, called the structure operator,*

$$D : \Omega(A, \mathcal{E}_n) \longrightarrow \Omega(A, \mathcal{E}_n)$$

which increases the total degree by one and satisfies $D^2 = 0$ and the graded derivation rule:

$$D(\omega\eta) = (d_A\omega)\eta + (-1)^k \omega D(\eta), \quad \forall\ \omega \in \Omega^k(A), \eta \in \Omega(A, \mathcal{E}). \tag{11}$$

We denote a representation up to homotopy of A by $(\mathcal{E}_n, D)$. Intuitively, a representation up to homotopy is a complex endowed with an A-connection which is "flat up to homotopy". It is the Lie algebroid version of Lada-Markl's L_∞-modules [LM95].

Proposition 2.9 [ACb] *There is a one-to-one correspondence between representation up to homotopy $(\mathcal{E}_n, D)$ of A and graded vector bundles $\mathcal{E}_n$ over M endowed with*

1. *A degree 1 operator ∂ on $\mathcal{E}_n$ making $(\mathcal{E}_n, \partial)$ a complex;*
2. *An A-connection ∇ on $(\mathcal{E}_n, \partial)$;*
3. *An $\mathrm{End}(\mathcal{E}_n)$ valued 2-form ω_2 of total degree 1, i.e. $\omega_2 \in \Omega^2(A, \mathrm{End}^{-1}(\mathcal{E}_n))$ satisfying*
$$\partial\omega_2 + R_\nabla = 0, \tag{12}$$
where R_∇ is the curvature of ∇.
4. *For each $i > 2$ an $\mathrm{End}(\mathcal{E}_n)$-valued i-form ω_i of total degree 1, i.e. $\omega_i \in \Omega^i(A, \mathrm{End}^{1-i}(\mathcal{E}_n))$ satisfying*
$$\partial\omega_i + d_\nabla\omega_i + \omega_2 \circ \omega_{i-2} + \cdots + \omega_{i-2} \circ \omega_2 = 0.$$

The correspondence is characterized by

$$D(\eta) = \partial\eta + d_\nabla\eta + \omega_2 \circ \eta + \omega_3 \circ \eta + \cdots .$$

We also write

$$D = \partial + d_\nabla + \omega_2 + \cdots .$$

We can **take the dual** of a representation up to homotopy $(\mathcal{E}_n, D)$. Consider the dual $\mathcal{E}_n^*[1]$, where the degree of E_i^* is $-i+1$. Then there is an operator $D^* : \Omega(A, \mathcal{E}_n^*) \longrightarrow \Omega(A, \mathcal{E}_n^*)$ uniquely determined by the condition

$$d_A(\eta \wedge \eta') = D^*(\eta) \wedge \eta' + (-1)^{|\eta|+1} \eta \wedge D(\eta'), \quad \forall\ \eta \in \Omega(A, \mathcal{E}_n^*), \eta' \in \Omega(A, \mathcal{E}_n), \tag{13}$$

where $\wedge$ is the operation $\Omega(A,\mathcal{E}^*)\otimes\Omega(A,\mathcal{E}_n)\longrightarrow\Omega(A)$ induced by the pairing between $\mathcal{E}_n^*$ and $\mathcal{E}_n$. Then $(\mathcal{E}_n^*,D^*)$ is a representation up to homotopy of A. In term of components of D, if $D=\partial+\nabla+\sum_{i\geq 2}\omega_i$, then we find that $D^*=\partial^*+\nabla^*+\sum_{i\geq 2}\omega_i^*$, where ∇^* is the connection dual to ∇ and,

$$\begin{aligned}\partial^*\eta_k &= -(-1)^k\eta_k\circ\partial,\\ \omega_p^*(X_1,\cdots,X_p)(\eta_k) &= -(-1)^{k(p+1)}\eta_k\circ\omega_p(X_1,\cdots,X_p),\end{aligned}$$

for any $\eta_k\in E_k^*$ and $X_1,\cdots,X_p\in\Gamma(A)$. For two representations up to homotopy, $(\mathcal{E}_n,D^{\mathcal{E}_n})$ and $(\mathcal{F}_m,D^{\mathcal{F}_m})$ of A, one can also take their **tensor product**. Consider the operator D corresponding to $\mathcal{E}_n\otimes\mathcal{F}_m$, which is uniquely determined by the condition

$$D(\eta_1\wedge\eta_2)=D^{\mathcal{E}_n}(\eta_1)\wedge\eta_2+(-1)^{|\eta_1|}\wedge D^{\mathcal{F}_m}(\eta_2),\quad\forall\ \eta_1\in\Omega(A,\mathcal{E}_n),\ \eta_2\in\Omega(A,\mathcal{F}_m).\quad(14)$$

Then $(\mathcal{E}_n\otimes\mathcal{F}_m,D)$ is a representation up to homotopy of A. In term of components, if $D^{\mathcal{E}_n}=\partial^{\mathcal{E}_n}+\nabla^{\mathcal{E}_n}+\sum_{i\geq 2}\omega_i^{\mathcal{E}_n}$ and $D^{\mathcal{F}_m}=\partial^{\mathcal{F}_m}+\nabla^{\mathcal{F}_m}+\sum_{i\geq 2}\omega_i^{\mathcal{F}_m}$, then $D=\partial+\nabla+\sum_{i\geq 2}\omega_i$, where

1. ∂ is the tensor product of $\partial^{\mathcal{E}_n}$ and $\partial^{\mathcal{F}_m}$:

$$\partial(u\otimes v)=\partial^{\mathcal{E}_n}u\otimes v+(-1)^{|u|}u\otimes\partial^{\mathcal{F}_m}v,\quad\forall\ u\in\mathcal{E}_n,v\in\mathcal{F}_m.$$

2. ∇ is the tensor product of $\nabla^{\mathcal{E}_n}$ and $\nabla^{\mathcal{F}_m}$:

$$\nabla_X(u\otimes v)=\nabla_X^{\mathcal{E}_n}u\otimes v+(-1)^{|u|}u\otimes\nabla_X^{\mathcal{F}_m}v,\quad\forall\ X\in\Gamma(A),u\in\Gamma(\mathcal{E}_n),v\in\Gamma(\mathcal{F}_m).$$

3. $\omega_p=\omega_p^{\mathcal{E}_n}\otimes\mathrm{Id}+\mathrm{Id}\otimes\omega_{\mathrm{p}}^{\mathcal{F}_{\mathrm{m}}}$.

Similarly, one can make the exterior algebra $\Lambda(\mathcal{E}_n^*[1])$ a representation up to homotopy with an operator $\widetilde{D}$, which is a derivation satisfying $\widetilde{D}^2=0$, on the algebra $\Omega(A,\Lambda(\mathcal{E}_n^*))=\Gamma(\Lambda(A^*))\otimes\Gamma(\Lambda(\mathcal{E}_n^*))$. Note that $\widetilde{D}$ is uniquely obtained from D^* by derivation extension.

2.3 Courant algebroids

A Courant algebroid is a vector bundle $E\longrightarrow M$ equipped with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the bundle, an antisymmetric bracket $[\![\cdot,\cdot]\!]$ on the section space $\Gamma(E)$ and a bundle map $\rho:E\longrightarrow TM$ such that a set of axioms are satisfied. A Courant algebroid is equivalent to a Lie 2-algebroid with a "degree-2 symplectic form" as proved in [Roy02].

We pay special attention to the exact Courant algebroid $(\mathcal{T}=TM\oplus T^*M,\langle\cdot,\cdot\rangle,[\![\cdot,\cdot]\!],\rho)$ associated to a manifold M with Ševera class $[H]$ with $H\in\Omega^3(M)$. Here the anchor $\rho:\mathcal{T}\longrightarrow TM$ is the projection. The canonical pairing $\langle\cdot,\cdot\rangle$ is given by

$$\langle X+\xi,Y+\eta\rangle=\frac{1}{2}\big(\xi(Y)+\eta(X)\big),\quad\forall\ X,Y\in\mathfrak{X}(M),\ \xi,\eta\in\Omega^1(M).\quad(15)$$

The antisymmetric bracket $[\![\cdot,\cdot]\!]$ is given by

$$[\![X+\xi,Y+\eta]\!]\triangleq[X,Y]+L_X\eta-L_Y\xi+\frac{1}{2}d(\xi(Y)-\eta(X))+i_{X\wedge Y}H.\quad(16)$$

It is not a Lie bracket, but we have

$$[\![[\![e_1, e_2]\!], e_3]\!] + c.p. = dT(e_1, e_2, e_3), \quad \forall\, e_1, e_2, e_3 \in \Gamma(\mathcal{T}), \tag{17}$$

where $T(e_1, e_2, e_3)$ is given by

$$T(e_1, e_2, e_3) = \frac{1}{3}(\langle [\![e_1, e_2]\!], e_3 \rangle + c.p.). \tag{18}$$

In this case, as shown in [Roy02], the corresponding Lie 2-algebroid is $T^*[2]T[1]M$ with the canonical symplectic form $dp_i \wedge dq^i + d\xi^i \wedge d\theta_i$ of a cotangent bundle and the homological vector field

$$Q = \xi^i \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial \theta_i} + \frac{1}{6} \frac{\partial \phi_{ijk}(q)}{\partial q^l} \xi^i \xi^j \xi^k \frac{\partial}{\partial p_l} - \frac{1}{6} \phi_{ijk}(q) \xi^i \xi^j \frac{\partial}{\partial \theta_k},$$

the Hamiltonian vector field of $\xi^i p_i - \frac{1}{6}\phi_{ijk}\xi^i\xi^j\xi^k$. The 3-form $H = \frac{1}{6}\phi_{ijk}\xi^i\xi^j\xi^k$. Here we must explain the local coordinates: q^i's are coordinates on M; $\xi^i = dq^i$ are cotangent vectors, thus coordinates on the fibre of $T[1]M$; then $p_i = \frac{\partial}{\partial q^i}$ and $\theta_i = \frac{\partial}{\partial \xi^i}$ are coordinates on the cotangent fibre. Then $(q^i, \xi^i, p_i, \theta_i)$ are of degree $0, 1, 2, 1$ respectively.

3 Semidirect products for representations up to homotopy

Lemma 3.1 *Given a Lie algebroid A and a graded vector bundle $\mathcal{E}_n = E_{-n+1} \oplus \cdots \oplus E_0$, let $\widetilde{D}$ be a representation up to homotopy of A on the exterior algebra $\Lambda(\mathcal{E}_n^*[1])$. Then the graded vector bundle $A \oplus \mathcal{E}_n$ with A of degree 0 is an NQ-manifold of degree n with homological vector field $\widetilde{D}$, thus a split Lie n-algebroid.*

Proof. A representation up to homotopy of A on the exterior algebra $\Lambda(\mathcal{E}_n^*[1])$ is a degree 1 operator $\widetilde{D}$ on

$$\Omega(A, \Lambda(\mathcal{E}_n^*[1])) = \Gamma(\Lambda(A^*[1])) \otimes \Gamma(\Lambda(\mathcal{E}_n^*[1])) = \Gamma(\Lambda(A \oplus \mathcal{E}_n)^*[1]) = C(A \oplus \mathcal{E}_n).$$

Then $\widetilde{D}$ serves as the degree 1 homological vector field. This makes $A \oplus \mathcal{E}$ an NQ-manifold of degree n. ■

Now we make the 2-term case more explicit. Given a graded vector bundle $\mathcal{E} = E_{-1} \oplus E_0$, a 2-term representation $(\mathcal{E}, D)$ of A is determined by [ACb, Remark 3.7]

1. a bundle map $\partial : E_{-1} \longrightarrow E_0$;
2. An A-connection ∇ on the complex $(\mathcal{E}, \partial)$. More precisely, there are A-connections on E_{-1} and E_0, which we denote by $\nabla^{E_{-1}}$ and ∇^{E_0} respectively, such that they are compatible with ∂: $\partial \circ \nabla_X^{E_{-1}} = \nabla_X^{E_0} \circ \partial$;
3. a 2-form $\omega \in \Omega^2(A, \mathrm{End}(E_0, E_{-1}))$ satisfying

$$R_{\nabla^{E_{-1}}} = \omega \circ \partial, \quad R_{\nabla^{E_0}} = \partial \circ \omega,$$

and $d_\nabla \omega = 0$, where R_∇ is the curvature.

In term of components, $D = \partial + \nabla + \omega$. As we stated in Section 2.2, there is an induced representation up to homotopy D^* on the shifted dual complex $\mathcal{E}^*[1]$ given by $D^* = \partial^* + \nabla^* - \omega^*$. Let $(\Lambda(\mathcal{E}^*[1]), \widetilde{D})$ be the corresponding representation up to homotopy on the exterior algebra $\Lambda(\mathcal{E}^*[1])$. By Lemma 3.1, we obtain a Lie 2-algebroid structure on $A \oplus \mathcal{E}$. We call this Lie 2-algebroid the **semidirect product** of the Lie algebroid A with its representation up to homotopy $(\mathcal{E}, D)$, and denote it by $A \ltimes \mathcal{E}$. We will give a more conceptual explanation in Section 4, where semidirect product corresponds to an extension with trivial extension class in $H^2(A, \mathcal{E})$.

The corresponding Chevalley-Eilenberg complex $C(A \ltimes \mathcal{E})$ is

$$C^\infty(M) \longrightarrow \Gamma((A \oplus E_0)^*) \longrightarrow \Gamma(\wedge^2(A \oplus E_0)^*) \oplus \Gamma(E_{-1}^*)$$
$$\longrightarrow \Gamma(\wedge^3(A \oplus E_0)^*) \oplus \Gamma((A \oplus E_0)^*) \otimes \Gamma(E_{-1}^*) \longrightarrow \cdots,$$

where E_0^* is of degree 1 and E_{-1}^* is of degree 2.

Now we make the brackets and anchor of the semidirect product more explicit.

Proposition 3.2 *Given a 2-term representation up to homotopy $(\mathcal{E} : E_{-1} \xrightarrow{\partial} E_0, \nabla, \omega)$ of A, the semidirect product Lie 2-algebroid structure on the graded vector bundle $[A \oplus E_0] \oplus E_{-1}$ is given by*

$$\rho(X + u) = \rho_A(X) \tag{19}$$
$$l_1(m) = \partial m, \tag{20}$$
$$l_2(X + u, Y + v) = [X, Y]_A + \nabla_X v - \nabla_Y u, \tag{21}$$
$$l_2(X + u, m) = \nabla_X m, \tag{22}$$
$$l_3(X + u, Y + v, Z + w) = \omega(X, Y)(w) + c.p., \tag{23}$$

for any $X, Y, Z \in \Gamma(A)$, $u, v, w \in \Gamma(E_0)$, $m \in \Gamma(E_{-1})$.

Proof. Let $D, D^*, \tilde{D}$ denote the same things as the above discussion. Now we apply the equations in (6) repetitively. For any $f \in C^\infty(M)$, we have

$$D(f) = \rho^*(df) = d_A f,$$

which implies that

$$\rho(X + u) = \rho_A(X).$$

Similarly, for any $\xi_0 \in \Gamma(E_0^*)$, $m \in E_{-1}$, we have

$$\langle D^*(\xi_0), m\rangle = \langle \partial^*(\xi_0), m\rangle = \langle \xi_0, -\partial m\rangle.$$

On the other hand, we have

$$\langle D^*(\xi_0), m\rangle = \langle \xi_0, -l_1(m)\rangle,$$

which implies that $l_1 = \partial$.

For any $\phi \in \Gamma(A^*) \otimes \Gamma(E_0^*)$, we have $D^*(\phi) = d_{\nabla^*}\phi$. Thus we have

$$\begin{aligned} D^*(\phi)(X,Y,u) &= \langle (d_{\nabla^*}\phi)(X,Y), u\rangle \\ &= \langle \nabla^*_X\phi(Y), u\rangle - \langle \nabla^*_Y\phi(X), u\rangle - \langle \phi([X,Y]_A), u\rangle \\ &= \rho(X)\langle \phi(Y), u\rangle - \langle \phi(Y), \nabla_X u\rangle - \rho(Y)\langle \phi(X), u\rangle + \langle \phi(X), \nabla_Y u\rangle \\ &\quad - \langle \phi([X,Y]_A), u\rangle \end{aligned}$$

On the other hand, by the relation among D^* and l_i and ρ, we have

$$\begin{aligned} D^*(\phi)(X,Y,u) &= \rho(X)\phi(Y,u) - \rho(Y)\phi(X,u) - \phi(l_2(X,Y),u) \\ &\quad -\phi(Y, l_2(X,u)) + \phi(X, l_2(Y,u)). \end{aligned}$$

Thus we have

$$\begin{aligned} l_2(X,Y) &= [X,Y]_A, & (24)\\ l_2(X,u) &= -l_2(u,X) = \nabla_X u. & (25) \end{aligned}$$

For any $\xi_1 \in \Gamma(E_1^*)$, we have

$$\langle D^*(\xi_1), X\otimes m\rangle = \langle d_{\nabla^*}\xi_1(X), m\rangle = \langle \nabla^*_X\xi_1, m\rangle = \rho(X)\langle \xi_1, m\rangle - \langle \xi_1, \nabla_X m\rangle .$$

On the other hand, by the relation among D^* and l_i and ρ, we have

$$\langle D^*(\xi_1), X\otimes m\rangle = \rho(X)\langle \xi_1, m\rangle - \langle \xi_1, l_2(X,m)\rangle .$$

Thus we have

$$l_2(X,m) = \nabla_X m. \qquad (26)$$

Furthermore, we have

$$\langle D^*(\xi_1), X\wedge Y\otimes u\rangle = \langle -\omega^*\circ\xi_1, X\wedge Y\otimes u\rangle = \langle -\omega^*(X,Y)(\xi_1), u\rangle = \langle \xi_1, \omega(X,Y)(u)\rangle .$$

On the other hand, we have

$$\langle D^*(\xi_1), X\wedge Y\otimes u\rangle = \langle \xi_1, l_3(X,Y,u)\rangle ,$$

which implies that

$$l_3(X,Y,u) = \omega(X,Y)(u).$$

Thus we have

$$l_3(X+u, Y+v, Z+w) = \omega(X,Y)(w) + c.p.. \qquad (27)$$

This finishes the proof. ■

Recall from [ACb, Example 3.28] that a Lie algebroid A has a natural coadjoint representation up to homotopy on $T^*M \xrightarrow{\rho^*} A^*$. Then any such two connections induce equivalent representations and equivalent representations give rise to isomorphic semidirect products. When $A = TM$, we arrive at the representation up to homotopy of TM on $T^*M \xrightarrow{\mathrm{Id}} T^*M$,

that is, a TM-connection ∇ on T^*M, and a 2-form $\omega \in \Omega^2(TM, \mathrm{End}(T^*M, T^*M))$ satisfying

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = \omega(X,Y), \quad \forall X, Y \in \Gamma(TM).$$

Then by Prop. 3.2, the Lie 2-algebroid structure on $TM \ltimes (T^*M \xrightarrow{\mathrm{Id}} T^*M)$ is given by

$$\begin{aligned}
\rho(X+u) &= \rho_A(X) && (28)\\
l_1(u) &= u, && (29)\\
l_2(X+u, Y+v) &= [X,Y] + \nabla_X(v) - \nabla_Y(u), && (30)\\
l_2(X+u, m) &= \nabla_X(m), && (31)\\
l_3(X+u, Y+v, Z+w) &= \omega(X,Y)(w) + c.p., && (32)
\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, $u, v, w, m \in \Gamma(T^*M)$. This is the case most essential to our purpose since it is used to construct the Courant algebroid. We explicitly write down the isomorphism between the semidirect products given by two representations up to homotopy (∇, ω) and (∇', ω') of TM. We assume that

$$\nabla_X - \nabla'_X = B(X)$$

for some bundle map $B : TM \longrightarrow \mathrm{Hom}(T^*M, T^*M)$. Denote by $\mathcal{M}$ and $\mathcal{M}'$ the corresponding semidirect product Lie 2-algebroids. Define $f = (f_0, f_1, f_2) : \mathcal{M} \longrightarrow \mathcal{M}'$ by

$$\begin{aligned}
f_0 &= \mathrm{Id} : TM \oplus T^*M \longrightarrow TM \oplus T^*M,\\
f_1 &= \mathrm{Id} : T^*M \longrightarrow T^*M,\\
f_2(X+\xi, Y+\eta) &= B(X)(\eta) - B(Y)(\xi).
\end{aligned}$$

In the following, we show that (f_0, f_1, f_2) is an isomorphism of Lie 2-algebroids by showing that it preserves the anchor and the brackets in an L_∞-fashion (see Remark 2.5). Notice that (f_0, f_1) is clearly an isomorphism of vector bundle complexes.

Since $f_0 = \mathrm{Id}$, the anchor ρ being the projection to TM is clearly preserved. Moreover, we have

$$\begin{aligned}
&\quad f_0 l_2(X+\xi, Y+\eta) - l'_2(f_0(X+\xi), f_0(Y+\eta))\\
&= \nabla_X \eta - \nabla_Y \xi - \left(\nabla'_X \eta - \nabla'_Y \xi\right)\\
&= B(X)(\eta) - B(Y)(\xi)\\
&= f_2(X+\xi, Y+\eta).
\end{aligned}$$

Similarly, we have

$$f_1 l_2(X+\xi, \eta) - l'_2(f_0(X+\xi), f_1(\eta)) \quad = \quad f_2(X+\xi, \eta).$$

Moreover, we have

$$\begin{aligned}
& l_2'(f_0(X+\xi), f_2(Y+\eta, Z+\gamma)) + c.p. - [f_2(l_2(X+\xi, Y+\eta), Z+\gamma) + c.p.] \\
= \;& l_2'(X+\xi, B(Y)(\gamma) - B(Z)(\eta)) + c.p. - [f_2([X,Y] + \nabla_X\eta - \nabla_Y\xi, Z+\gamma) + c.p.] \\
= \;& \nabla'_X(B(Y)(\gamma) - B(Z)(\eta)) + c.p. \\
& -[B([X,Y])(\gamma) + B(Z)(\nabla_X\eta - \nabla_Y\xi) + c.p.] \\
= \;& -\nabla'_X\nabla'_Y\gamma + \nabla'_Y\nabla'_X\gamma + \nabla'_{[X,Y]}\gamma + c.p. \\
& +\nabla_X\nabla_Y\gamma - \nabla_Y\nabla_X\gamma - \nabla_{[X,Y]}\gamma + c.p. \\
= \;& -\omega'(X,Y)(\gamma) + c.p. + [\omega(X,Y)(\gamma) + c.p.] \\
= \;& -l_3'(f_0(X+\xi), f_0(Y+\eta), f_0(Z+\gamma)) + f_1 l_3(X+\xi, Y+\eta, Z+\gamma).
\end{aligned}$$

Thus (f_0, f_1, f_2) is an isomorphism of Lie 2-algebroids. Therefore, if we only care about the isomorphism class, we can take *the* semidirect product $TM \ltimes (T^*M \xrightarrow{\mathrm{Id}} T^*M)$.

Theorem 3.3 *Let $\mathcal{E} := E_{-1} \oplus E_0$ be a 2-term graded vector bundle over M. Suppose that there is a Lie 2-algebroid structure on $(A \oplus E_0) \oplus E_{-1}$ given by a degree 1 homological vector field Q,*

$$\begin{aligned}
& C^\infty(M) \xrightarrow{Q} \Gamma((A\oplus E_0)^*) \xrightarrow{Q} \Gamma(\wedge^2(A\oplus E_0)^*) \oplus \Gamma(E_{-1}^*) \\
& \xrightarrow{Q} \Gamma(\wedge^3(A\oplus E_0)^*) \oplus \Gamma((A\oplus E_0)^*) \otimes \Gamma(E_{-1}^*) \xrightarrow{Q} \cdots .
\end{aligned}$$

Then this Lie 2-algebroid structure is the semidirect product $A \ltimes \mathcal{E}$ if and only if the following conditions are satisfied:

(1). *for any k, the restriction of Q on $\Gamma(\wedge^k A^*)$ is exactly given by d_A, i.e.*

$$Q|_{\Gamma(\wedge^k A^*)} = d_A : \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^{k+1} A^*);$$

(2). $Q(\Gamma(E_0^*)) \subset \Gamma(E_{-1}^*) \oplus \Gamma(A^* \otimes E_0^*)$.

(3). $Q(\Gamma(E_{-1}^*)) \subset \Gamma(A^* \otimes E_{-1}^*) \oplus \Gamma(\wedge^2 A^* \otimes E_0^*)$.

Proof. The semidirect product $A \ltimes \mathcal{E}$ obviously satisfies the three conditions.

Conversely, denote by Q_0 and Q_1 the components of $Q(\Gamma(E_0^*))$ in $\Gamma(E_{-1}^*)$ and $\Gamma(A^* \otimes E_0^*)$, since Q is a derivation of degree 1 and $Q|_{\Gamma(\wedge^k A^*)} = d_A$, we obtain that Q_0 is given by a bundle map ∂^* and Q_1 is given by $d_{\nabla_0^*}$ for some A-connection ∇_0^* on E_0^*. Similarly, the components of $Q(E_{-1}^*)$ in $\Gamma(A^* \otimes E_1^*)$ and $\Gamma(\wedge^2 A^* \otimes E_0^*)$ are determined by $d_{\nabla_1^*}$ and $\omega^* \in \Gamma(\wedge^2 A^* \otimes \mathrm{End}(E_{-1}^*, E_0^*))$, where ∇_1^* is an A-connection on E_{-1}^*.

Let $\partial : E_{-1} \longrightarrow E_0$ be the dual map of ∂^*, ∇_0 and ∇_1 be the dual connection on E_0 and E_{-1} of ∇_0^* and ∇_1^*, and $\omega \in \Gamma(\wedge^2 A^* \otimes \mathrm{End}(E_0, E_{-1}))$ be given by

$$\omega(X,Y)(u_0)(\xi_1) = -\langle \omega^*(X,Y)(\xi_1), u_0 \rangle .$$

Then $Q^2 = 0$ implies that $D = \partial + d_\nabla + \omega$ is a degree 1 operation on $\Omega(A; E)$ satisfying $D^2 = 0$ and graded derivation rule, i.e. $(\mathcal{E}, D)$ is a representation up to homotopy of A. By

Leibniz rule, Q is totally determined by D which implies that the Lie 2-algebroid structure corresponding to Q is the semidirect product $A \ltimes \mathcal{E}$. ■

The symplectic NQ-manifold associated to the standard Courant algebroid $TM \oplus T^*M$ is $T^*[2]T[1]M$. The symplectic structure is the standard one: $dp_i \wedge dq^i + d\xi^i \wedge d\theta_i$ and the degree 1 vector field Q is given by

$$Q = \xi^i \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial \theta_i}. \tag{33}$$

In the following, we show that for the standard Courant algebroid, the degree 1 vector field Q given by (33) satisfies the conditions given in Theorem 3.3.

Theorem 3.4 *The standard Courant algebroid $T^*[2]T[1]M$ is isomorphic to the semidirect product $TM \ltimes (T^*M \xrightarrow{\mathrm{Id}} T^*M)$ as NQ manifolds.*

Proof. By Theorem 3.3, we only need to show that the vector field Q given by (33) satisfies conditions (1)-(3) listed in Theorem 3.3. For any $f \in C^\infty(M)$, we have

$$Q(f) = \xi^i \frac{\partial f}{\partial q^i} = df.$$

For any $\xi = f_j \xi^j \in \Gamma(T^*[1]M)$, with $f_j \in C^\infty(M)$, we have

$$Q(\xi) = \xi^i \frac{\partial \xi}{\partial q^i} = \frac{\partial f_j}{\partial q^i} \xi^i \xi^j = d\xi \in \Omega^2(M).$$

Then the fact that Q satisfies Condition (1) in Theorem 3.3 follows from the derivation property of Q.

For any $\theta = f^j \theta_j \in \Gamma(T[1]M)$, with $f^j \in C^\infty(M)$, we have

$$Q(\theta) = \xi^i \frac{\partial \theta}{\partial q^i} + p_i \frac{\partial \theta}{\partial \theta_i} = \xi^i \frac{\partial f^j}{\partial q^i} \theta_j + f^j p_i \frac{\partial \theta_j}{\partial \theta_i} = \frac{\partial f^j}{\partial q^i} \xi^i \theta_j + f^i p_i,$$

which implies that Condition (2) in Theorem 3.3 is satisfied.

Finally for any $p = f^j p_j \in \Gamma(T^*[2]M)$, we have

$$Q(p) = \xi^i \frac{\partial p}{\partial q^i} = \frac{\partial f^j}{\partial q^i} \xi^i p_j.$$

Thus Condition (3) in Theorem 3.3 is also satisfied. ■

Remark 3.5 *In local coordinates $(q^i, \xi^i, p_i, \theta_i)$, if we choose a TM-connection ∇ on T^*M by*

$$\nabla_{\theta_i} \xi^j = 0. \tag{34}$$

Then it is flat, i.e. $\omega = 0$. It is straightforward to see that in this case, $D = \partial + d_\nabla + \omega = \mathrm{Id} + d_\nabla$ is exactly given by $\xi^i \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial \theta_i}$.

4 Extension of Lie 2-algebroids

Now we consider a very specific extension of Lie 2-algebroids in the following form:

$$
\begin{array}{ccccccccc}
0 & \xrightarrow{0} & E_{-1} & \xrightarrow{\mathrm{Id}} & \widehat{A}_{-1} = E_{-1} & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \\
{\scriptstyle 0}\downarrow & & {\scriptstyle \partial}\downarrow & & {\scriptstyle \widehat{\partial}}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow \\
0 & \xrightarrow{0} & E_0 & \xrightarrow{\mathbb{i}_0} & \widehat{A}_0 & \xrightarrow{\mathbb{p}} & A & \xrightarrow{0} & 0
\end{array} \tag{35}
$$

The exact sequence consists of two exact sequences of vector bundles with all squares commutative diagrams of vector bundles over the same base M. Moreover, A is a Lie algebroid viewed as the Lie 2-algebroid $0 \xrightarrow{0} A$ as in Example 2.6, $E_{-1} \xrightarrow{\partial} E_0$ is a 2-term complex of vector bundles viewed as an abelian Lie 2-algebroid as in Example 2.7, $(\widehat{A}_{-1} \xrightarrow{\widehat{\partial}} \widehat{A}_0, \hat{\rho}, \hat{l}_2, \hat{l}_3)$ is a Lie 2-algebroid, and $(\mathrm{Id}, \mathbb{i}_0)$ and $(0, \mathbb{p})$ are strict morphisms (see Def. 2.4) over id_M between Lie 2-algebroids. Moreover we assume that $\mathbb{p}$ admits a global splitting. We call this sort of extension a *2-term abelian extension of a Lie algebroid A*. An *isomorphism* between 2-term abelian extensions of A is a morphism between the two corresponding exact sequences (35) such that its restriction on A and $E_\bullet$ are identities, and its restriction on the middle term is an isomorphism of Lie 2-algebroids (not necessarily strict, see Def. 2.4 and Remark 2.5).

Recall that there is a 1-1 correspondence between Lie algebra extensions of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{a}$ and $H^2(\mathfrak{g}, \mathfrak{a})$ (which includes the information of $\mathfrak{a}$ as a $\mathfrak{g}$ representation). Now we establish a similar correspondence for the abelian extensions of a Lie algebroid by a 2-term representation up to homotopy.

Given an extension (35), since $\mathbb{p}$ admits a splitting, say $\sigma : A \to \widehat{A}_0$, we can write $\widehat{A}_0 = A \oplus E_0$. Since the diagram is commutative, we have that $\hat{\partial} = 0 + \partial$, $\hat{\rho} = \rho_A$ and $\mathbb{i}_0$ is the inclusion map, which we usually omit. Define an A-connection ∇ on the complex $\mathcal{E}$ and $\omega : \wedge^2 A \longrightarrow \mathrm{End}(E_0, E_{-1})$ by

$$
\begin{aligned}
\nabla_X(u) &= \hat{l}_2(\sigma(X), u), \\
\nabla_X(m) &= \hat{l}_2(\sigma(X), m), \\
\omega(X,Y)(u) &= \hat{l}_3(\sigma(X), \sigma(Y), u),
\end{aligned}
$$

for any $X, Y \in A,\ u \in E_0, m \in E_{-1}$.

Lemma 4.1 *With the above notations, (∇, ω) gives a representation up to homotopy of the Lie algebroid A on the complex $\mathcal{E} : E_{-1} \xrightarrow{\partial} E_0$.*

Proof. It is not hard to deduce that

$$
\begin{aligned}
& [\nabla_X, \nabla_Y](u) - \nabla_{[X,Y]_A} u \\
= \ & \hat{l}_2(\sigma(X), \hat{l}_2(\sigma(Y), u)) - \hat{l}_2(\sigma(Y), \hat{l}_2(\sigma(X), u)) - \hat{l}_2(\sigma([X,Y]_A), u) \\
= \ & \hat{l}_2(\sigma(X), \hat{l}_2(\sigma(Y), u)) - \hat{l}_2(\sigma(Y), \hat{l}_2(\sigma(X), u)) \\
& - \hat{l}_2(\hat{l}_2(\sigma(X), \sigma(Y)), u) + \hat{l}_2(c_2(X,Y), u).
\end{aligned}
$$

Thus we have

$$[\nabla_X, \nabla_Y](u) - \nabla_{[X,Y]_A} u = \hat{\partial}\hat{l}_3(\sigma(X), \sigma(Y), u) = \partial(\omega(X,Y)(u)).$$

Similarly, we have

$$[\nabla_X, \nabla_Y](m) - \nabla_{[X,Y]_A} m = \hat{l}_3(\sigma(X), \sigma(Y), \partial m) = \omega(X,Y)(\partial m).$$

By definition, we have

$$\begin{aligned}
[\nabla_X, \omega(Y,Z)](u) &= \nabla_X \omega(Y,Z)(u) - \omega(Y,Z)(\nabla_X u) \\
&= \hat{l}_2(\sigma(X), \hat{l}_3(\sigma(Y), \sigma(Z), u)) - \hat{l}_3(\sigma(Y), \sigma(Z), \hat{l}_2(\sigma(X), u)), \\
\omega([X,Y]_A, Z)(u) &= \hat{l}_3(\sigma([X,Y]_A), \sigma(Z), u) \\
&= \hat{l}_3(\hat{l}_2(\sigma(X), \sigma(Y)) - c_2(X,Y), \sigma(Z), u) \\
&= \hat{l}_3(\hat{l}_2(\sigma(X), \sigma(Y)), \sigma(Z), u).
\end{aligned}$$

Since we have

$$\hat{l}_2(\hat{l}_3(\sigma(X), \sigma(Y), \sigma(Z)), u) = 0,$$

by the Jacobiator identity of $\hat{l}_3$, we deduce that

$$\begin{aligned}
& [\nabla_X, \omega(Y,Z)](u) + c.p. - (\omega([X,Y]_A, Z)(u) + c.p.) \\
= \ & \hat{l}_2(\sigma(X), \hat{l}_3(\sigma(Y), \sigma(Z), u)) - \hat{l}_3(\hat{l}_2(\sigma(X), \sigma(Y)), \sigma(Z), u) + c.p. \\
= \ & 0,
\end{aligned}$$

which implies that (∇, ω) is a representation up to homotopy of A on $E_{-1} \xrightarrow{\partial} E_0$. ■

Now we recall how to define the cohomology $H^\bullet(A, \mathcal{E})$ for a 2-term representation up to homotopy $(\mathcal{E} = E_{-1} \oplus E_0, D)$ of a Lie algebroid A. Such a representation up to homotopy gives us a complex:

$$\begin{aligned}
& E_{-1} \xrightarrow{D} E_0 \oplus \mathrm{Hom}(A, E_{-1}) \xrightarrow{D} \mathrm{Hom}(A, E_0) \oplus \mathrm{Hom}(\wedge^2 A, E_{-1}) \xrightarrow{D} \\
& \mathrm{Hom}(\wedge^2 A, E_0) \oplus \mathrm{Hom}(\wedge^3 A, E_{-1}) \xrightarrow{D} \mathrm{Hom}(\wedge^3 A, E_0) \oplus \mathrm{Hom}(\wedge^4 A, E_{-1}) \longrightarrow \cdots,
\end{aligned} \tag{36}$$

where $\mathrm{Hom}(\wedge^k A, E_i) := \Gamma(\wedge^k A^* \otimes E_i)$. We write $D = \partial + d_\nabla + \omega$ according to Prop. 2.9. Then for any k-cochain $(\varpi_1, \varpi_2) \in \mathrm{Hom}(\wedge^k A, E_0) \oplus \mathrm{Hom}(\wedge^{k+1} A, E_{-1})$, we have

$$D(\varpi_1, \varpi_2) = (d_\nabla \varpi_1 + (-1)^{k+1} \partial \circ \varpi_2, d_\nabla \varpi_2 + (-1)^{k+1} \omega \circ \varpi_1). \tag{37}$$

This complex eventually gives us the cohomology $H^\bullet(A, \mathcal{E})$ with coefficient in the representation up to homotopy $\mathcal{E}$.

In particular, for a 2-cochain $(c_2, c_3) \in \mathrm{Hom}(\wedge^2 A, V_0) \oplus \mathrm{Hom}(\wedge^3 A, V_{-1})$, we have

$$D(c_2, c_3) = (d_\nabla c_2 - \partial \circ c_3) + (d_\nabla c_3 - \omega \circ c_2) \in \mathrm{Hom}(\wedge^3 A, V_0) \oplus \mathrm{Hom}(\wedge^4 A, V_{-1}).$$

Thus (c_2, c_3) is a 2-cocycle means that

$$d_\nabla c_2 - \partial \circ c_3 = 0, \quad d_\nabla c_3 - \omega \circ c_2 = 0.$$

Now for our extension, we define $c_2 \in \mathrm{Hom}(\wedge^2 A, E_0)$ and $c_3 \in \mathrm{Hom}(\wedge^3 A, E_{-1})$ by

$$c_2(X,Y)=\hat{l}_2(\sigma(X),\sigma(Y))-\sigma([X,Y]_A),\quad c_3(X,Y,Z)=\hat{l}_3(\sigma(X),\sigma(Y),\sigma(Z)).$$

Transfer the Lie 2-algebroid structure $(\widehat{A}_{-1}\xrightarrow{\hat{l}_1=\widehat{\partial}}\widehat{A}_0,\hat{\rho},\hat{l}_2,\hat{l}_3)$ to the complex $E_{-1}\xrightarrow{0+\partial} A\oplus E_0$, we obtain a Lie 2-algebroid $(E_{-1}\xrightarrow{0+\partial} A\oplus E_0,\rho,l_2,l_3)$, where

$$\begin{aligned}
\rho(X+u)&=\rho_A(X),\\
l_2(X+u,Y+v)&=[X,Y]_A+\nabla_X v-\nabla_Y u+c_2(X,Y),\\
l_2(X+u,m)&=\nabla_X m,\\
l_3(X+u,Y+v,Z+w)&=c_3(X,Y,Z)+\omega(X,Y)(w)+\omega(Y,Z)(u)+\omega(Z,X)(v).
\end{aligned}\tag{38}$$

Then we have,

Lemma 4.2 *With the above notation, (c_2,c_3) is a 2-cocycle of the Lie algebroid A with coefficient the representation up to homotopy $(E_{-1}\xrightarrow{\partial}E_0,\nabla,\omega)$ (see the complex (36)). Conversely, given a 2-term representation up to homotopy $(E_{-1}\xrightarrow{\partial}E_0,\nabla,\omega)$ of A and a 2-cocycle $(c_2,c_3)\in \mathrm{Hom}(\wedge^2A,E_0)\oplus\mathrm{Hom}(\wedge^3A,E_{-1})$, define ρ,l_2,l_3 by (38), then we obtain a Lie 2-algebroid*

$$A\ltimes_{(c_2,c_3)}\mathcal{E}:=(E_{-1}\xrightarrow{0+\partial}A\oplus E_0,\rho,l_2,l_3)$$

which is a 2-term abelian extension of A and fits inside (35).

Proof. On one hand, by direct computations, we have

$$\begin{aligned}
&l_2(Z+w,l_2(X+u,Y+v))+c.p.\\
=\;&l_2(Z+w,[X,Y]_A+\nabla_X v-\nabla_Y u+c_2(X,Y))+c.p.\\
=\;&-\nabla_{[X,Y]_A}w+\nabla_Z\nabla_X v-\nabla_Z\nabla_Y u+\nabla_Z c_2(X,Y)\\
&+c_2(Z,[X,Y]_A)+c.p.\\
=\;&d_\nabla c_2(X,Y,Z)+\partial(\omega(X,Y)(w)+\omega(Y,Z)(u)+\omega(Z,X)(v)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&l_2(Z+w,l_2(X+u,Y+v))+c.p.\\
=\;&\partial l_3(Z+w,X+u,Y+v)\\
=\;&\partial c_3(X,Y,Z)+\partial(\omega(X,Y)(w)+\omega(Y,Z)(u)+\omega(Z,X)(v)).
\end{aligned}$$

Thus we have

$$d_\nabla c_2-\partial c_3=0.\tag{39}$$

By (7), we have

$$l_2(X+u,l_3(Y+v,Z+w,P+x))+c.p.=l_3(l_2(X+u,Y+v),Z+w,P+x)+c.p.,$$

which implies that

$$\nabla_X c_3(Y,Z,P)+\nabla_X(\omega(Y,Z)(x)+\omega(Z,P)(v)+\omega(P,Y)(w))+c.p.$$

is equal to

$$\begin{aligned}&c_3([X,Y]_A,Z,P)+\omega(Z,P)c_2(X,Y)+\omega(Z,P)(\nabla_X v-\nabla_Y u)\\&+\omega([X,Y]_A,Z)(x)+\omega(P,[X,Y]_A)(w)+c.p..\end{aligned}$$

By the fact that (∇,ω) is a representation up to homotopy, we deduce that

$$d_\nabla c_3-\omega\circ c_2=0. \tag{40}$$

By (39) and (40), we deduce that (c_2,c_3) is a 2-cocycle. It is straightforward to check the converse part. ■

Denote by $\widetilde{c_2}:\Gamma(E_0^*)\longrightarrow\Gamma(\wedge^2A^*)$ and $\widetilde{c_3}:\Gamma(E_{-1}^*)\longrightarrow\Gamma(\wedge^3A^*)$ the dual of c_2 and c_3 respectively, i.e.

$$\widetilde{c_2}(\xi)(X,Y)=-\xi(c_2(X,Y)),\quad \widetilde{c_3}(\sigma)(X,Y,Z)=\sigma(c_3(X,Y,Z)), \tag{41}$$

for any $\xi\in\Gamma(E_0^*),\sigma\in\Gamma(E_{-1}^*)$ and $X,Y,Z\in\Gamma(A)$. We denote their graded derivation extension on $C(A\oplus\mathcal{E})$ using the same notation (see (42)). Then, we have

Proposition 4.3 *The Chevalley-Eilenberg complex $C(A\oplus\mathcal{E})$ associated to $A\ltimes_{(c_2,c_3)}\mathcal{E}$ with the Lie 2-algebroid structure (38) is given by*

$$\begin{aligned}&C^\infty(M)\xrightarrow{\widehat{D}}\Gamma((A\oplus E_0)^*)\xrightarrow{\widehat{D}}\Gamma(\wedge^2(A\oplus E_0)^*)\oplus\Gamma(E_{-1}^*)\\&\xrightarrow{\widehat{D}}\Gamma(\wedge^3(A\oplus E_0)^*)\oplus\Gamma((A\oplus E_0)^*)\otimes\Gamma(E_{-1}^*)\xrightarrow{\widehat{D}}\cdots,\end{aligned} \tag{42}$$

in which $\Gamma(E_0^*)$ is of degree 1 and $\Gamma(E_{-1}^*)$ is of degree 2, and $\widehat{D}$ is given by

$$\widehat{D}=\widetilde{D}+\widetilde{c_2}+\widetilde{c_3},$$

with $(\Lambda(\mathcal{E}^*[1]),\widetilde{D})$ the induced representation up to homotopy on the exterior algebra.

Proof. For any $\xi\in\Gamma(E_0^*)$ and $X,Y\in\Gamma(A)$, we have

$$\widehat{D}(\xi)(X,Y)=\langle\xi,-l_2(X,Y)\rangle=\langle\xi,-c_2(X,Y)\rangle,$$

which implies that

$$\widehat{D}(\xi)(X,Y)=\widetilde{c_2}(\xi)(X,Y).$$

Similarly, for any $\kappa\in E_{-1}^*$, we have

$$\widehat{D}(\kappa)(X,Y,Z)=\langle\kappa,l_3(X,Y,Z)\rangle=\langle\kappa,c_3(X,Y,Z)\rangle.$$

Thus we have

$$\widehat{D}(\kappa)(X,Y,Z)=\widetilde{c_3}(\kappa)(X,Y,Z).$$

Therefore, we have

$$\widehat{D}=\widetilde{D}+\widetilde{c_2}+\widetilde{c_3}.$$

Since $\widehat{D}$ is given by a Lie 2-algebroid structure, it must satisfy $\widehat{D}^2=0$. ■

Give a representation up to homotopy $(\mathcal{E},\nabla,\omega)$ and a 2-cocycle (c_2,c_3), by Lemma 4.2, we can construct an extension of Lie 2-algebroids $A\ltimes_{(c_2,c_3)}\mathcal{E}$. We further prove that the extension does not depend on the cocycle itself but on its cohomology class in $H^2(A,\mathcal{E})$.

Proposition 4.4 *If two 2-cocycles (c_2, c_3) and (c_2', c_3') represent the same cohomology class in $H^2(A, \mathcal{E})$, then the corresponding extensions $A \ltimes_{(c_2,c_3)} \mathcal{E}$ and $A \ltimes_{(c_2',c_3')} \mathcal{E}$ are isomorphic. Conversely, if extensions $A \ltimes_{(c_2,c_3)} \mathcal{E}$ and $A \ltimes_{(c_2',c_3')} \mathcal{E}$ are isomorphic, then (c_2, c_3) and (c_2', c_3') represent the same cohomology class provided $\partial : E_{-1} \longrightarrow E_0$ is injective or surjective, or $c_2 = 0$.*

Proof. Assume that 2-cocycles (c_2, c_3) and (c_2', c_3') represent the same cohomology, then we have $(c_2, c_3) = (c_2', c_3') + D(e_1, e_2)$, for some $(e_1, e_2) \in \mathrm{Hom}(A, E_0) \oplus \mathrm{Hom}(\wedge^2 A, E_{-1})$, more precisely,

$$c_2 = c_2' + d_\nabla e_1 + \partial e_2, \quad c_3 = c_3' + d_\nabla e_2 + \omega \circ e_1.$$

Define $(f_0, f_1) : A \ltimes_{(c_2,c_3)} \mathcal{E} \longrightarrow A \ltimes_{(c_2',c_3')} \mathcal{E}$ by

$$\begin{aligned} f_0(X+u) &= X + u + e_1(X), \\ f_1(m) &= m, \end{aligned}$$

and define $f_2 : \wedge^2(A \oplus E_0) \longrightarrow E_{-1}$ by

$$f_2(X+u, Y+v) = e_2(X, Y).$$

It is clear that (f_0, f_1) is an isomorphism of complexes of vector bundles. Thus we only need to show that (f_0, f_1, f_2) preserves the anchor and the brackets in an L_∞-fashion (see Remark 2.5). First we have

$$\rho(X+u) = \rho_A(X) = \rho'(X + u + e_1(X)) = \rho'(f_0(X+u)),$$

which implies that f_0 preserves the anchor. Moreover, we have

$$\begin{aligned} & l_2'(f_0(X+u), f_0(Y+v)) \\ = \ & l_2'(X + u + e_1(X), Y + v + e_1(Y)) \\ = \ & [X, Y]_A + \nabla_X(v) - \nabla_Y(u) + \nabla_X(e_1(Y)) - \nabla_Y(e_1(X)) + c_2'(X, Y), \end{aligned}$$

and

$$f_0 l_2(X+u, Y+v) = [X, Y]_A + \nabla_X(v) - \nabla_Y(u) + e_1([X, Y]_A) + c_2(X, Y).$$

Thus we have

$$\begin{aligned} f_0 l_2(X+u, Y+v) - l_2'(f_0(X+u), f_0(Y+v)) &= -d_\nabla e_1(X, Y) + (c_2 - c_2')(X, Y) \\ &= \partial e_2(X, Y) \\ &= \partial f_2(X+u, Y+v). \end{aligned}$$

Similarly, we have

$$\begin{aligned} f_1 l_2(X+u, m) - l_2'(f_0(X+u), f_1(m)) &= \nabla_X(m) - \nabla_X(m) = 0 \\ &= f_2(X+u, \partial m). \end{aligned}$$

By a computation, we have

$$\begin{aligned}
& l_2'(f_0(X+u), f_2(Y+v, Z+w)) + c.p. - \big(f_2(l_2(X+u, Y+v), Z+w) + c.p.\big) \\
= \; & l_2'(X+u+e_1(X), e_2(Y,Z)) + c.p. \\
& -\big(f_2([X,Y]_A + \nabla_X(v) - \nabla_Y(u) + c_2(X,Y), Z+w) + c.p.\big) \\
= \; & \nabla_X e_2(Y,Z) - c.p. - \big(e_2([X,Y]_A, Z) + c.p.\big) \\
= \; & d_\nabla e_2(X,Y,Z).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& f_1 l_3(X+u, Y+v, Z+w) - l_3'(f_0(X+u), f_0(Y+v), f_0(Z+w)) \\
= \; & c_3(X,Y,Z) - c_3'(X,Y,Z) - (\omega(X,Y)(e_1(Z)) + c.p.) \\
= \; & d_\nabla e_2(X,Y,Z).
\end{aligned}$$

Thus (f_0, f_1, f_2) is an isomorphism from the Lie 2-algebroid $A \ltimes_{(c_2,c_3)} \mathcal{E}$ to $A \ltimes_{(c_2',c_3')} \mathcal{E}$. Furthermore, it is obvious that the corresponding extensions are also isomorphic.

Conversely, given two 2-cocycles (c_2, c_3) and (c_2', c_3'), let (f_0, f_1, f_2) be an isomorphism of the resulting extensions, we can assume that

$$f_0(X+u) = X + e_1(X) + u, \quad f_1(m) = m,$$

for some $e_1 \in \mathrm{Hom}(A, E_0)$. By computation, we have

$$\begin{aligned}
& l_2'(f_0(X+u), f_0(Y+v)) \\
= \; & [X,Y]_A + \nabla_X e_1(Y) + \nabla_X v - \nabla_Y e_1(X) - \nabla_Y u + c_2'(X,Y), \\
& f_0(l_2(X+u, Y+v)) \\
= \; & [X,Y]_A + e_1([X,Y]_A) + \nabla_X v - \nabla_Y u + c_2(X,Y).
\end{aligned}$$

By (8), we obtain

$$c_2(X,Y) - c_2'(X,Y) = (d_\nabla e_1)(X,Y) + \partial f_2(X,Y), \tag{43}$$

$$\partial f_2(X,v) = 0. \tag{44}$$

Similarly, by (9), we get

$$f_2(X, \partial m) = 0, \quad \forall\, X \in \Gamma(A),\ m \in \Gamma(E_{-1}). \tag{45}$$

Furthermore, we have

$$\begin{aligned}
l_3'(f_0(X+u), f_0(Y+v), f_0(Z+w)) &= c_3'(X,Y,Z) + \big(\omega(X,Y)(e_1(Z)+w) + c.p.\big), \\
f_1(l_3(X+u, Y+v, Z+w)) &= c_3(X,Y,Z) + \big(\omega(X,Y)(w) + c.p.\big).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& l_3'(f_0(X+u), f_0(Y+v), f_0(Z+w)) - f_1(l_3(X+u, Y+v, Z+w)) \\
& = c_3'(X,Y,Z) - c_3(X,Y,Z) + \big(\omega(X,Y)(e_1(Z)) + c.p.\big).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& l_2'(f_0(X+u), f_2(Y+v, Z+w)) + c.p. - \big(f_2(l_2(X+u, Y+v), Z+w) + c.p.\big) \\
= \;& d_\nabla f_2(X,Y,Z) - \big(f_2(c_2(X,Y),Z) + c.p.\big) \\
& + \nabla_X (f_2(Y,w) + f_2(v,Z) + f_2(v,w)) + c.p. \\
& - f_2([X,Y]_A + c_2(X,Y), w) + c.p. - f_2(\nabla_X v - \nabla_Y u, Z+w) + c.p.
\end{aligned}$$

Thus (10) is equivalent to

$$\begin{aligned}
(c_3 - c_3')(X,Y,Z) &= d_\nabla f_2(X,Y,Z) + \omega(X,Y)(e_1(Z)) - f_2(c_2(X,Y),Z) + c.p., && (46) \\
\nabla_X f_2(v,w) &= f_2(\nabla_X v, w) + f_2(v, \nabla_X w) && (47)
\end{aligned}$$

and

$$\nabla_X f_2(Y,w) + \nabla_Y f_2(w,X) = f_2([X,Y]_A + c_2(X,Y), w) - f_2(\nabla_Y w, X) + f_2(\nabla_X w, Y). \quad (48)$$

By (44) and (45), if ∂ is injective, or surjective, we have

$$f_2(X,u) = 0, \quad \forall\, X \in \Gamma(A),\ u \in \Gamma(E_0),$$

which implies that

$$f_2(c_2(X,Y),Z) + c.p. = 0.$$

Thus, if ∂ is injective, or surjective, or $c_2 = 0$, define $e_2 \in \Gamma(\mathrm{Hom}(\wedge^2 A, E_{-1}))$ by $e_2(X,Y) = f_2(X,Y)$. By (43) and (46), we have

$$(c_2, c_3) = (c_2', c_3') + D(e_1, e_2).$$

Therefore, (c_2, c_3) and (c_2', c_3') are in the same cohomology class. ■

Thus with Lemma 4.1, Lemma 4.2, Prop. 4.3, and Prop. 4.4, we have the following conclusion

Theorem 4.5 *Given a Lie algebroid A and its 2-term representation up to homotopy $\mathcal{E}$, the isomorphism classes of the abelian extensions of A by $\mathcal{E}$ are classified by $H^2(A,\mathcal{E})$ provided that $\partial : E_{-1} \to E_0$ is injective or surjective.*

Remark 4.6 *Unlike in the classical case, we do not have a full classification result essentially because f_2 controls the deficiency of bracket-preserving only through ∂. It is the case even if the representation is trivial, namely in the central extension case. Unfortunately, it is not easy to come up with a counter example neither, because f_2 needs to satisfy (47) and (48). In fact, L_∞-algebras and (non-strict) L_∞-morphisms do not form a model category and do not have desired limits and colimits. This is probably the reason where the proof breaks down when immitating the classical one[4].*

We call the element in $H^2(A,\mathcal{E})$ corresponding to an extension, the *extension class* of this extension. When the extension class is 0, the extension is given by the semidirect product in the last section. Thus we also have

[4] Private talk to Bruno Vallette.

Theorem 4.7 *Given a Lie algebroid A and its 2-term representation up to homotopy $\mathcal{E}$, an abelian extension of A by $\mathcal{E}$ is isomorphic to the semidirect product $A \ltimes \mathcal{E}$ if and only if its extension class in $H^2(A,\mathcal{E})$ is trivial.*

Similarly to Theorem 3.3, we have

Theorem 4.8 *Let $\mathcal{E} = E_{-1} \oplus E_0$ be a 2-term graded vector bundle over M and A be a Lie algebroid over M. Suppose that there is a Lie 2-algebroid structure on $(A \oplus E_0) \oplus E_{-1}$ given by a degree 1 homological vector field Q,*

$$\begin{aligned} &C^\infty(M) \xrightarrow{Q} \Gamma((A\oplus E_0)^*) \xrightarrow{Q} \Gamma(\wedge^2(A\oplus E_0)^*) \oplus \Gamma(E_{-1}^*) \\ &\xrightarrow{Q} \Gamma(\wedge^3(A\oplus E_0)^*) \oplus \Gamma((A\oplus E_0)^*) \otimes \Gamma(E_{-1}^*) \xrightarrow{Q} \cdots . \end{aligned}$$

Then this Lie 2-algebroid is the abelian extension of A with the extension class in $H^2(A,\mathcal{E})$ represented by the cocycle (c_2, c_3) if and only if the following conditions are satisfied:

(1). *for any k, the restriction of Q on $\Gamma(\wedge^k A^*)$ is exactly given by d_A, i.e.*

$$Q|_{\Gamma(\wedge^k A^*)} = d_A : \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^{k+1} A^*).$$

(2). $Q(\Gamma(E_0^*)) \subset \Gamma(E_{-1}^*) \oplus \Gamma(A^* \otimes E_0^*) \oplus \Gamma(\wedge^2 A^*)$.

(3). $Q(\Gamma(E_{-1}^*)) \subset \Gamma(A^* \otimes E_{-1}^*) \oplus \Gamma(\wedge^2 A^* \otimes E_0^*) \oplus \Gamma(\wedge^3 A^*)$.

Moreover, we have a twisted version of Theorem 3.4 (which is in the case of $(c_2, c_3) = 0$):

Proposition 4.9 *The exact Courant algebroid $T^*[2]T[1]M$ with Ševera class $[H] \in H^3(M,\mathbb{R})$ is isomorphic to the extension of TM by the coadjoint representation up to homotopy $(T^*M \xrightarrow{\mathrm{Id}} T^*M, \nabla, \omega)$ with the extension class*

$$(c_2, c_3) \in \Gamma(\mathrm{Hom}(\wedge^2 TM, T^*M) \oplus \mathrm{Hom}(\wedge^3 TM, T^*M))$$

given by

$$\begin{aligned} c_2(X,Y) &= i_{X\wedge Y} H, && (49)\\ c_3(X,Y,Z) &= d_\nabla c_2(X,Y,Z) = \nabla_X c_2(Y,Z) + c.p. - (c_2([X,Y],Z) + c.p.). && (50) \end{aligned}$$

Proof. The extension Lie 2-algebroid structure is given by

$$\begin{aligned} l_2(X+\xi, Y+\eta) &= [X,Y] + \nabla_X \eta - \nabla_Y \xi + i_{X\wedge Y} H, \\ l_2(X+\xi, \eta) &= \nabla_X \eta, \\ l_3(X+\xi, Y+\eta, Z+\gamma) &= d_\nabla c_2(X,Y,Z) + \omega(X,Y)(\gamma) + c.p. \end{aligned}$$

It fits into the following exact sequence of Lie 2-algebroids,

$$\begin{array}{ccccccccc} 0 & \xrightarrow{0} & T^*M & \xrightarrow{\mathrm{Id}} & T^*M & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \\ {\scriptstyle 0}\downarrow & & {\scriptstyle \mathrm{Id}}\downarrow & & {\scriptstyle 0+\mathrm{Id}}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow \\ 0 & \xrightarrow{0} & T^*M & \xrightarrow{\mathbb{i}_0} & TM\oplus T^*M & \xrightarrow{\mathbb{p}} & TM & \xrightarrow{0} & 0. \end{array} \qquad (51)$$

To see that it is isomorphic to the exact Courant algebroid with Ševera class $[H]$, we only need to show that in local coordinates, their degree 1 homological vector field are same. Take the same local coordinates as in Section 2.3 and choose the connection given by (34). By remark 3.5, we only need to show that in these coordinates,

$$\widetilde{c_2} + \widetilde{c_3} = \frac{1}{6}\frac{\partial \phi_{ijk}(q)}{\partial q^l}\xi^i\xi^j\xi^k\frac{\partial}{\partial p_l} - \frac{1}{6}\phi_{ijk}(q)\xi^i\xi^j\frac{\partial}{\partial \theta_k},$$

where c_2 and c_3 are given by (49) and (50) respectively. In fact, by (41), we have

$$\widetilde{c_2}(\theta_k)(\theta_i,\theta_j) = -\theta_k(c_2(\theta_i,\theta_j)) = -\frac{1}{6}\phi_{ijk}(q),$$

which implies that

$$\widetilde{c_2} = -\frac{1}{6}\phi_{ijk}(q)\xi^i\xi^j\frac{\partial}{\partial \theta_k}.$$

Now we have that

$$c_3 = d_\nabla c_2 = \frac{1}{6}\frac{\partial \phi_{ijk}(q)}{\partial q^l}\xi^i\xi^j\xi^k dq^l,$$

thus by (41), we have

$$\widetilde{c_3}(p_l)(\theta_i,\theta_j,\theta_k) = p_l(c_3(\theta_i,\theta_j,\theta_k)) = \frac{1}{6}\frac{\partial \phi_{ijk}(q)}{\partial q^l},$$

which implies that

$$\widetilde{c_3} = \frac{1}{6}\frac{\partial \phi_{ijk}(q)}{\partial q^l}\xi^i\xi^j\xi^k\frac{\partial}{\partial p_l}.$$

This completes the proof. ■

However we have the following lemma:

Lemma 4.10 *The cohomology group $H^k(TM, T^*M \xrightarrow{\mathrm{Id}} T^*M) = 0$ for all $k \geq 1$.*

Proof. Take a k-cocycle $(c_k, c_{k+1}) \in \mathrm{Hom}(\wedge^k TM, T^*M) \oplus \mathrm{Hom}(\wedge^{k+1} TM, T^*M)$. Then $D(c_k, c_{k+1}) = 0$ implies that $d_\nabla c_k = (-1)^k c_{k+1}$. Thus $(c_k, c_{k+1}) = D(0, (-1)^k c_k)$ is a coboundary. ■

Thus with Thm. 4.7, Prop. 4.9, and the above lemma, we conclude that,

Corollary 4.11 *The exact Courant algebroid $T^*[2]T[1]M$ with Ševera class $[H]$ is isomorphic as NQ manifolds to the standard Courant algebroid with 0 Ševera class.*

We do have an example of non-trivial extension.

Example 4.12 (String Lie 2-algebras) *A string Lie 2-algebra is a 2-term L_∞-algebra $\hat{\mathfrak{g}}$ with $\hat{\mathfrak{g}}_0 = \mathfrak{g}$ a semisimple Lie algebra of compact type, $\hat{\mathfrak{g}}_{-1} = \mathbb{R}$, and*

$$\begin{aligned} \partial = 0, \quad l_2((e_1,r_1),(e_2,r_2)) &= ([e_1,e_2],0), \\ l_3((e_1,r_1),(e_2,r_2),(e_3,r_3)) &= (0, \langle [e_1,e_2],e_3\rangle_{Killing}), \end{aligned}$$

where $e_1,\ e_2,\ e_3 \in \mathfrak{g},\ r_1,\ r_2,\ r_3 \in \mathbb{R}$. *Then it fits into the following extension*

$$
\begin{array}{ccccccccc}
0 & \xrightarrow{0} & \mathbb{R} & \xrightarrow{\mathrm{Id}} & \hat{\mathfrak{g}}_{-1} & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \\
{\scriptstyle 0}\big\downarrow & & {\scriptstyle \partial}\big\downarrow & & {\scriptstyle \hat{\partial}}\big\downarrow & & {\scriptstyle 0}\big\downarrow & & {\scriptstyle 0}\big\downarrow \\
0 & \xrightarrow{0} & 0 & \xrightarrow{\mathbb{i}_0} & \hat{\mathfrak{g}}_0 = \mathfrak{g} & \xrightarrow{\mathbb{p}} & \mathfrak{g} & \xrightarrow{0} & 0,
\end{array} \tag{52}
$$

with the extension class represented by $(0, c_3)$ *with* $c_3 : \wedge^3\mathfrak{g} \to \mathbb{R}$ *given by* $c_3(e_1, e_2, e_3) = \langle [e_1, e_2], e_3\rangle_{Killing}$. *The class represented by* $(0, c_3)$ *is non-zero in* $H^2(\mathfrak{g}, \mathbb{R} \to 0)$ *because the class represented by* c_3 *in* $H^3(\mathfrak{g}, \mathbb{R})$ *is known to be non-zero (actually it is the generator of* $H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$*).*

5 Integration

We now integrate an abelian extension of a Lie algebroid A by a 2-term representation up to homotopy, $(\mathcal{E}, D)$, with the extension class represented by a 2-cocycle, (c_2, c_3). The general idea is that we first integrate the representation up to homotopy $(\mathcal{E}, D)$ to a representation up to homotopy $(\mathcal{E}, F_1, F_2)$ of the fundamental Lie groupoid $\mathcal{G}$ of A. Then we integrate the extension class (c_2, c_3) into a groupoid extension class (C_2, C_3). Then we use F_1, F_2 and (C_2, C_3) to construct the extension Lie 2-groupoid and take it as the integration of the extension Lie 2-algebroid. Notice that both of the above two integration processes have obstruction (see [ASb, Prop. 5.4], [ASa, Thm. 4.7]). Here we use this general idea more as a guideline to construct the integration object of a Courant algebroid.

In the case of Courant algebroids, the Lie algebroid $A = TM$ is the tangent bundle. Thus the fundamental groupoid $\mathcal{G}$ is the usual fundamental groupoid $\Pi_1(M) = \tilde{M} \times \tilde{M}/\pi_1(M) \rightrightarrows M$, where $\tilde{M}$ is the simply connected cover of M. When M is simply connected, $\Pi_1(M) = M \times M$ is simply the pair groupoid. Then the representation up to homotopy of TM on $T^*M \xrightarrow{\mathrm{Id}} T^*M$ is the coadjoint representation up to homotopy. By [ACb, Theorem 3.10], any such two representations up to homotopy are equivalent. Thus we assume that the coadjoint representation of TM integrate to a coadjoint representation up to homotopy $(T^*M \xrightarrow{\mathrm{Id}} T^*M, F_1, F_2)$ of $\Pi_1(M)$.

5.1 Preliminaries

Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a Lie groupoid. We denote the space of sequences $(g_1, \cdots, g_k)$ of composable arrows (i.e. $t(g_i) = s(g_{i-1})$) in $\mathcal{G}$ by $\mathcal{G}_k$.

Definition 5.1 [ACa] *A unital* 2*-term representation up to homotopy of a Lie groupoid consists of*

1. *A* 2*-term complex of vector bundles over* G_0*:* $E_{-1} \xrightarrow{\partial} E_0$.

2. *A nonassociative representation* F_1 *on* E_0 *and* E_{-1} *satisfying*

$$\partial \circ F_1 = F_1 \circ \partial, \quad F_1(1_G) = \mathrm{Id}.$$

3. *A smooth map* $F_2 : \mathcal{G}_2 \longrightarrow \mathrm{End}(V_0, V_{-1})$ *such that*

$$F_1(g_1) \cdot F_1(g_2) - F_1(g_1 g_2) = [\partial, F_2(g_1, g_2)], \tag{53}$$

as well as

$$F_1(g_1) \circ F_2(g_2, g_3) - F_2(g_1 g_2, g_3) + F_2(g_1, g_2 g_3) - F_2(g_1, g_2) \circ F_1(g_3) = 0. \tag{54}$$

Given such a representation up to homotopy $(\mathcal{E}, F_1, F_2)$, we can also define a complex to compute the cohomology $H^\bullet(\mathcal{G}, \mathcal{E})$ as in the case of usual representation (see [ASa, Prop.2.9]). Here we recall the formula in our case of 2-term representation: the complex is

$$C(\mathcal{G}, \mathcal{E})^n = \oplus_{n=k+l} C^k(\mathcal{G}, E_l), \quad \text{where } C^k(\mathcal{G}, E_l) = Maps(\mathcal{G}_k, t^* E_l).$$

The differential D is given by

$$D = \widetilde{\partial} + \widetilde{F}_1 + \widetilde{F}_2,$$

where given $\eta \in C^k(\mathcal{G}, E_l)$,

$$\begin{aligned} \widetilde{\partial}(\eta) &= \partial \circ \eta, \\ \widetilde{F}_1(\eta)(g_1, \cdots, g_{k+1}) &= (-1)^{k+l} \Big(F_1(g_1) \eta(g_2, \cdots, g_{k+1}) \\ &\quad + \sum_{i=1}^{p} (-1)^i \eta(g_1, \cdots, g_i g_{i+1}, \cdots, g_{k+1}) + (-1)^{k+1} \eta(g_1, \cdots, g_k) \Big), \end{aligned}$$

and

$$\widetilde{F}_2(\eta)(g_1, \cdots, g_{k+2}) = F_2(g_1, g_2)(\eta(g_3, \cdots, g_{k+2})).$$

In the case of 2-term representation $E_{-1} \xrightarrow{\partial} E_0$, a 2-cochain is made up by two terms $(C_2, C_3) \in C^2(G, E_0) \oplus C^3(G, E_{-1})$. The cocycle conditions read

$$\widetilde{F}_1(C_2) + \partial \circ C_3 = 0, \tag{55}$$

$$\widetilde{F}_1(C_3) + \widetilde{F}_2(C_2) = 0. \tag{56}$$

Throughout in this paper, unless specifically mentioned, all cocycles are normalized, that is,

$$\eta(g_1, \ldots, g_k) = 0, \quad \text{if one of } g_1, \ldots, g_k \text{ is } 1_x \text{ for some } x \in G_0.$$

5.2 Extensions of Lie groupoids

First we recall a classical fact: given a representation V of a group G, and a 2-cocycle $C \in C^2(G, V)$, there is a group extension

$$1 \to V \to \widehat{G} \to G \to 1,$$

where V is viewed as an abelian group with multiplication the addition of its vector space structure. When the 2-cocycle is trivial, $\widehat{G}$ is isomorphic to the semidirect product $G \ltimes V$.

We would like to establish a similar theory in the Lie 2-groupoid case and show that the integration of Courant algebroid is such an extension Lie 2-groupoid (because Courant algebroid itself can be realized as an extension Lie 2-algebroid).

The concept of Lie n-groupoid is best and uniformly given via Kan complexes. However, to describe a Lie 2-groupoid, there is another method, which is much longer to write down, but easier to understand as a comparison with Lie groupoids. A Lie 2-groupoid is a groupoid object in the 2-category **GpdBibd** where the space of objects is only a manifold (but not a general Lie groupoid). Here **GpdBibd** is the 2-category with Lie groupoids as objects, Hilsum-Skandalis (HS) bimodules as morphisms, isomorphisms of HS bimodules as 2-morphisms. Thus the category of manifolds embeds into this 2-category by viewing a manifold M as a trivial groupoid $M \rightrightarrows M$ which only has identity arrows. The equivalence of such two descriptions is given in [Zhu09]. A special sort of Lie 2-groupoid is a groupoid object in the 2-category of **Gpd** with the space of object a manifold, where **Gpd** is a sub-2-category of **GpdBibd** containing only strict groupoid morphisms as morphisms. We call such Lie 2-groupoid semistrict Lie 2-groupoid. Since the Lie 2-groupoid integrating Courant algebroid that we construct is an example of semistrict Lie 2-groupoids, we describe this concept explicitly below.

Definition 5.2 *A semistrict Lie 2-groupoid consists of:*

- *a smooth manifold G_0, which is the set of objects $x, y, z, \cdots$,*
- *a smooth manifold G_1, which is the set of 1-morphisms $g, h, \cdots$. For a 1-morphism $g : x \longrightarrow y$, we write $\alpha(g) = x$, $\beta(g) = y$. For another 1-morphism $h : y \longrightarrow z$, we write their composition as $hg : x \longrightarrow z$.*
- *a Lie groupoid $\mathcal{G} : G_2 \rightrightarrows G_1$, where G_2 is the set of 2-morphisms ϕ, $\phi' \cdots$. For any 2-morphism $\phi : g \Longrightarrow h$, where $g, h : x \longrightarrow y$ are 1-morphisms, the source and target maps s, t are given by $s(\phi) = g$, $t(\phi) = h$. The composition in this groupoid is usually called vertical multiplication, and denoted by $\cdot_{\mathrm{v}}$. We require $\beta s = \beta t$ and $\alpha s = \alpha t$.*
- *For all objects $x, y, z \in G_0$, there is a Lie groupoid morphism $\mathcal{G} \times_{\alpha s, G_0, \beta s} \mathcal{G} \to \mathcal{G}$, which is called horizontal multiplication and denoted by $\cdot_{\mathrm{h}}$, i.e. for $\phi : g \Longrightarrow h : x \longrightarrow y$ and $\phi' : g' \Longrightarrow h' : y \longrightarrow z$, we have*

  $$\phi' \cdot_{\mathrm{h}} \phi : g'g \Longrightarrow h'h : x \longrightarrow z,$$

  or, in terms of a diagram,

  $$\begin{tikzcd} z & y \arrow[l, bend right=40, "g'"'] \arrow[l, bend left=40, "h'"] \arrow[l, phantom, "\Downarrow\phi'"] & \cdot_{\mathrm{h}} & y & x \arrow[l, bend right=40, "g"'] \arrow[l, bend left=40, "h"] \arrow[l, phantom, "\Downarrow\phi"] & = & z & x. \arrow[l, bend right=40, "g'g"'] \arrow[l, bend left=40, "h'h"] \arrow[l, phantom, "\Downarrow\phi' \cdot_{\mathrm{h}} \phi"] \end{tikzcd}$$

- *for any $x \in G_0$, there is an identity 1-morphism and an identity 2-morphism, which we both denote by 1_x.*
- *a Lie groupoid contravariant morphism* $\mathrm{inv} : \mathcal{G} \to \mathcal{G}$,

and the following natural isomorphisms

- *the associator $a_{(g_1, g_2, g_3)} : (g_1 \cdot_{\mathrm{h}} g_2) \cdot_{\mathrm{h}} g_3 \longrightarrow g_1 \cdot_{\mathrm{h}} (g_2 \cdot_{\mathrm{h}} g_3)$.*

- *the left and right unit* $l_g : 1_{\beta(g)} \cdot_{\mathrm{h}} g \longrightarrow g$ *and* $r_g : g \cdot_{\mathrm{h}} 1_{\alpha(g)} \longrightarrow g$*,*
- *the unit and counit* $i_g : 1_{\beta(g)} \longrightarrow g \cdot_{\mathrm{h}} \mathrm{inv}(g)$ *and* $e_g : \mathrm{inv}(g) \cdot_{\mathrm{h}} g \longrightarrow 1_{\alpha(g)}$

which are such that the following diagrams commute:

- *the* **pentagon identity** *for the associator*

$$
\begin{tikzcd}
 & (g_1 \cdot_{\mathrm{h}} g_2) \cdot_{\mathrm{h}} (g_3 \cdot_{\mathrm{h}} g_4) \arrow[rr, "a_{g_1,g_2,g_3 \cdot_{\mathrm{h}} g_4}"] & & Q \\
((g_1 \cdot_{\mathrm{h}} g_2) \cdot_{\mathrm{h}} g_3) \cdot_{\mathrm{h}} g_4 \arrow[ur, "a_{(g_1 \cdot_{\mathrm{h}} g_2),g_3,g_4}"] \arrow[dr, "a_{g_1,g_2,g_3} \cdot_{\mathrm{h}} 1_{g_4}"'] & & & \\
 & (g_1 \cdot_{\mathrm{h}} (g_2 \cdot_{\mathrm{h}} g_3)) \cdot_{\mathrm{h}} g_4 \arrow[r, "a_{g_1,g_2 \cdot_{\mathrm{h}} g_3,g_4}"] & g_1 \cdot_{\mathrm{h}} ((g_2 \cdot_{\mathrm{h}} g_3) \cdot_{\mathrm{h}} g_4) \arrow[uur, "1_{g_1} \cdot_{\mathrm{h}} a_{g_2,g_3,g_4}"'] &
\end{tikzcd}
$$

  where $Q = g_1 \cdot_{\mathrm{h}} (g_2 \cdot_{\mathrm{h}} (g_3 \cdot_{\mathrm{h}} g_4))$*.*

- *the* **triangle identity** *for the left and right unit lows:*

$$
\begin{tikzcd}
(g_1 \cdot_{\mathrm{h}} 1_{\alpha(g_1)}) \cdot_{\mathrm{h}} g_2 \arrow[rr, "a_{g_1,1_{\alpha(g_1)},g_2}"] \arrow[dr, "r_{g_1} \cdot_{\mathrm{h}} 1_{g_2}"'] & & g_1 \cdot_{\mathrm{h}} (1_{\alpha(g_1)} \cdot_{\mathrm{h}} g_2) \arrow[dl, "1_{g_1} \cdot_{\mathrm{h}} l_{g_2}"] \\
 & g_1 \cdot_{\mathrm{h}} g_2 &
\end{tikzcd}
$$

- *the* **first zig-zag identity***:*

$$
\begin{tikzcd}
 & (g \cdot_{\mathrm{h}} \mathrm{inv}(g)) \cdot_{\mathrm{h}} g \arrow[r, "a_{g,\mathrm{inv}(g),g}"] & g \cdot_{\mathrm{h}} (\mathrm{inv}(g) \cdot_{\mathrm{h}} g) \arrow[dr, "1_g \cdot_{\mathrm{h}} e_g"] & \\
1_{\beta(g)} \cdot_{\mathrm{h}} g \arrow[ur, "i_g \cdot_{\mathrm{h}} 1_g"] \arrow[drr, "l_g"'] & & & g \cdot_{\mathrm{h}} 1_{\alpha(g)} \\
 & & g \arrow[ur, "r_g^{-1}"'] &
\end{tikzcd}
$$

- *the* **second zig-zag identity***:*

$$
\begin{tikzcd}
 & \mathrm{inv}(g) \cdot_{\mathrm{h}} (g \cdot_{\mathrm{h}} \mathrm{inv}(g)) \arrow[r, "a_{\mathrm{inv}(g),g,\mathrm{inv}(g)}"] & (\mathrm{inv}(g) \cdot_{\mathrm{h}} g) \cdot_{\mathrm{h}} \mathrm{inv}(g) \arrow[dr, "e_g \cdot_{\mathrm{h}} 1_{\mathrm{inv}(g)}"] & \\
\mathrm{inv}(g) \cdot_{\mathrm{h}} 1_{\beta(g)} \arrow[ur, "1_{\mathrm{inv}(g)} \cdot_{\mathrm{h}} i_g"] \arrow[drr, "r_{\mathrm{inv}(g)}"'] & & & 1_{\alpha(g)} \cdot_{\mathrm{h}} \mathrm{inv}(g). \\
 & & \mathrm{inv}(g) \arrow[ur, "l^{-1}_{\mathrm{inv}(g)}"'] &
\end{tikzcd}
$$

For simplicity, we denote a semistrict Lie 2-groupoid by $G_2 \rightrightarrows G_1 \rightrightarrows G_0$. In the special case where a_{g_1,g_2,g_3}, l_g, r_g, i_g, e_g are all identity isomorphisms, we call such a Lie 2-groupoid a *strict Lie 2-groupoid.* If G_0 is a point, we obtain the concept of a *semistrict Lie 2-group* [BL04, SZb].

For any vector bundle E, it can be viewed as an abelian Lie groupoid with the source and the target both the projection to the base and multiplication the pointwise addition. Similarly, we have

Example 5.3 *Any 2-term complex of vector bundles $\mathcal{E} : E_{-1} \xrightarrow{\partial} E_0$ has an "abelian" strict Lie 2-groupoid structure, which we denote by $E_\bullet$. First, we have an action groupoid $E_0 \rtimes_{G_0} E_{-1} \rightrightarrows E_0$ where E_{-1} acts on E_0 by*

$$u \cdot m = u + \partial m.$$

Furthermore, the pointwise addition of E_i gives horizontal multiplication of the Lie 2-groupoid, that is

$$(u, m) \cdot_{\mathrm{h}} (v, n) = (u + v, m + n). \tag{57}$$

Moreover $inv(u, m) = (-u, -m)$, $1_p = (0_p, 0_p)$ for a point p on the base.

An *abelian extension* of a Lie groupoid $\mathcal{G}$ by a 2-term representation $\mathcal{E}$ is a short exact sequence of Lie 2-groupoids with the left term $E_\bullet$ viewed as an abelian Lie 2-groupoid. The next step we explain what an exact sequence of Lie 2-groupoid is (but only in our very special case)[5]. In our special case, all the Lie 2-groupoid morphisms we mention here are strict, namely they respect all the structure maps strictly without further 2-morphisms. Given a Lie 2-groupoid $\hat{G}_\bullet$ and a Lie groupoid $\mathcal{G}$ with $\hat{G}_0 = G_0$, a 2-groupoid morphism $\phi_\bullet : \hat{G}_\bullet \to \mathcal{G}$ with $\phi_0 = id_{G_0}$ is surjective if ϕ_1 is a surjective submersion (then this implies that ϕ_2 is a surjective submersion). Given another Lie 2-groupoid $\hat{H}_\bullet$ with $\hat{H}_0 = \hat{G}_0$, a 2-groupoid morphism $\iota_\bullet : \hat{H}_\bullet \to \hat{G}_\bullet$ with $\iota_0 = id_{G_0}$ is injective if ι_1 and ι_2 are embeddings. The image $\mathrm{im}(\iota_\bullet)$ is naturally defined as the image 2-groupoid under $\iota_\bullet$. The kernel $\ker(\phi_\bullet)$ is made up by G_0, and the subsets of $\hat{G}_1$ and $\hat{G}_2$ which maps to $\{1_x, x \in G_0\}$ under $\phi_\bullet$. Since the identity of $G_\bullet$ is strict and $\phi_{1,2}$ are surjective submersions, $\ker(\phi_\bullet)$ is a Lie 2-groupoid. We call the short sequence $\hat{H}_\bullet \xrightarrow{\iota_\bullet} \hat{G}_\bullet \xrightarrow{\phi_\bullet} \mathcal{G}$ exact if $\iota_\bullet$ is injective, $\phi_\bullet$ is surjective, and $\ker(\phi_\bullet) = \mathrm{im}(\iota_\bullet)$ as Lie 2-groupoids.

$$\begin{tikzcd}
 & H_2 \arrow[r, "\iota_2"] \arrow[d, Rightarrow] & \hat{G}_2 \arrow[r, "\phi_2"] \arrow[d, Rightarrow] & G_1 \arrow[d, Rightarrow] & \\
1 \arrow[r] & H_1 \arrow[r, "\iota_1"] \arrow[d, Rightarrow] & \hat{G}_1 \arrow[r, "\phi_1"] \arrow[d, Rightarrow] & G_1 \arrow[r] \arrow[d, Rightarrow] & 1. \\
 & H_0 = G_0 \arrow[r, "\iota_0"] & \hat{G}_0 = G_0 \arrow[r, "\phi_0"] & G_0 &
\end{tikzcd}$$

Even though this definition is very restrictive, it includes the following example, which is the most important one for our purpose of integration. We have the following proposition which can be viewed as the global version of Lemma 4.2: (however, we shall not expect a classification result as in Thm. 4.7 with our current version of groupoid cohomology because even in the case of group this version needs to be refined for the classification result to hold. See [WZ]).

Proposition 5.4 *Given a 2-term representation up to homotopy $(\mathcal{E}, F_1, F_2)$ of a Lie groupoid $\mathcal{G} = G_1 \rightrightarrows G_0$ and a 2-cocycle $(C_2, C_3) \in C^2(\mathcal{G}, \mathcal{E})$, there is a Lie 2-groupoid structure on*

[5]There should be a more general notation of exact sequence of Lie 2-groupoids using generalized morphisms allowing higher morphisms. It should include a Kan fibration as an example. On the other hand, our definition here is a special case of Kan fibration.

$G_1 \times_{G_0} E_0 \times_{G_0} E_{-1} \rightrightarrows G_1 \times_{G_0} E_0 \rightrightarrows G_0$ which is an extension of $\mathcal{G}$ and fits into the exact sequence

$$\begin{tikzcd}
 & E_0 \times_{G_0} E_{-1} \arrow[r, "\iota_2"] \arrow[d, shift left] \arrow[d, shift right] & G_1 \times_{G_0} E_0 \times_{G_0} E_{-1} \arrow[r, "\phi_2"] \arrow[d, shift left] \arrow[d, shift right] & G_1 \arrow[d, shift left] \arrow[d, shift right] & \\
1 \arrow[r] & E_0 \arrow[r, "\iota_1"] \arrow[d, shift left] \arrow[d, shift right] & G_1 \times_{G_0} E_0 \arrow[r, "\phi_1"] \arrow[d, shift left] \arrow[d, shift right] & G_1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 1 \\
 & G_0 \arrow[r, "\iota_0 = id_{G_0}"] & G_0 \arrow[r, "\phi_0 = id_{G_0}"] & G_0 &
\end{tikzcd}$$

with natural inclusion $\iota_\bullet$ and natural projection $\phi_\bullet$. The Lie 2-groupoid structure of the left term is abelian as in Example 5.3. The Lie 2-groupoid structure of the middle term is semistrict and given by the following data: The source map s and target map t are given by

$$s(g,\xi,m) = (g,\xi), \quad t(g,\xi,m) = (g,\xi + \partial m), \tag{58}$$

and α, β (see the second item of Def. 5.2) are given by

$$\alpha(g,\xi) = \alpha(g), \quad \beta(g,\xi) = \beta(g),$$

for any $(g,\xi) \in G_1 \times_{G_0} E_0$.

The vertical multiplication $\cdot_{\mathrm{v}}$ is given by

$$(h,\eta,n) \cdot_{\mathrm{v}} (g,\xi,m) = (g,\xi,m+n), \quad \textit{where } h = g, \eta = \xi + \partial m.$$

The horizontal multiplication $\cdot_{\mathrm{h}}$ of objects is given by

$$(g_1,\xi) \cdot_{\mathrm{h}} (g_2,\eta) = (g_1 g_2, \xi + F_1(g_1)(\eta) + C_2(g_1,g_2)), \tag{59}$$

the horizontal multiplication $\cdot_{\mathrm{h}}$ of morphisms is given by

$$(g_1,\xi,m) \cdot_{\mathrm{h}} (g_2,\eta,n) = (g_1 g_2, \xi + F_1(g_1)(\eta) + C_2(g_1,g_2), m + F_1(g_1)(n)). \tag{60}$$

The associator

$$a_{(g_1,\xi),(g_2,\eta),(g_3,\gamma)} : \big((g_1,\xi) \cdot_{\mathrm{h}} (g_2,\eta)\big) \cdot_{\mathrm{h}} (g_3,\gamma) \longrightarrow (g_1,\xi) \cdot_{\mathrm{h}} \big((g_2,\eta) \cdot_{\mathrm{h}} (g_3,\gamma)\big)$$

is given by

$$\begin{aligned} &a_{(g_1,\xi),(g_2,\eta),(g_3,\gamma)} = \\ &(g_1g_2g_3, \xi + F_1(g_1)(\eta) + F_1(g_1g_2)(\gamma) + C_2(g_1,g_2) + C_2(g_1g_2,g_3), F_2(g_1,g_2)(\gamma) - C_3(g_1,g_2,g_3)). \end{aligned} \tag{61}$$

The inverse map inv *is given by*

$$\mathrm{inv}(g,\xi) = (g^{-1}, -F_1(g^{-1})(\xi) - C_2(g^{-1},g)), \tag{62}$$

and

$$\mathrm{inv}(g,\xi,m) = (g^{-1}, -F_1(g^{-1})(\xi) - C_2(g^{-1},g), -F_1(g^{-1})(m)). \tag{63}$$

The identity 1-morphisms are $(1_x,0)$ and the identity 2-morphisms are $(1_x,0,0)$. The unit $i_{(g,\xi)} : (1_{\beta(g)},0) \longrightarrow (g,\xi) \cdot_{\mathrm{h}} \mathrm{inv}(g,\xi)$ is given by

$$i_{(g,\xi)} = (1_{\beta(g)}, 0, -F_2(g,g^{-1})(\xi) + C_3(g,g^{-1},g)). \tag{64}$$

All the other natural isomorphisms are identity isomorphisms.

Proof. First we verify the Lie 2-groupoid structure of the middle term. The verification is similar to the proof of [SZb, Thm. 3.7]. By (58), (59) and (60), it is straightforward to see that

$$\begin{aligned} s\big((g_1,\xi,m)\cdot_{\mathrm{h}}(g_2,\eta,n)\big) &= s(g_1,\xi,m)\cdot_{\mathrm{h}} s(g_2,\eta,n),\\ t\big((g_1,\xi,m)\cdot_{\mathrm{h}}(g_2,\eta,n)\big) &= t(g_1,\xi,m)\cdot_{\mathrm{h}} t(g_2,\eta,n), \end{aligned}$$

and

$$\begin{aligned} &\big((g,\xi+\partial m,n)\cdot_{\mathrm{h}}(g',\eta+\partial p,q)\big)\cdot_{\mathrm{v}}\big((g,\xi,m)\cdot_{\mathrm{h}}(g',\eta,p)\big)\\ =\ &\big((g,\xi+\partial m,n)\cdot_{\mathrm{v}}(g,\xi,m)\big)\cdot_{\mathrm{h}}\big((g',\eta+\partial p,q)\cdot_{\mathrm{v}}(g',\eta,p)\big). \end{aligned}$$

This implies that the multiplication $\cdot_{\mathrm{h}}$ is a groupoid morphism.

We compute that

$$\begin{aligned} &\big((g_1,\xi)\cdot_{\mathrm{h}}(g_2,\eta)\big)\cdot_{\mathrm{h}}(g_3,\gamma)\\ =\ &(g_1g_2,\xi+F_1(g_1)(\eta)+C_2(g_1,g_2))\cdot_{\mathrm{h}}(g_3,\gamma)\\ =\ &(g_1g_2g_3,\xi+F_1(g_1)(\eta)+C_2(g_1,g_2)+F_1(g_1g_2)(\gamma)+C_2(g_1g_2,g_3)),\\ &(g_1,\xi)\cdot_{\mathrm{h}}\big((g_2,\eta)\cdot_{\mathrm{h}}(g_3,\gamma)\big)\\ =\ &(g_1,\xi)\cdot_{\mathrm{h}}(g_2g_3,\eta+F_1(g_2)(\gamma)+C_2(g_2,g_3))\\ =\ &\big(g_1g_2g_3,\xi+F_1(g_1)(\eta+F_1(g_2)(\gamma)+C_2(g_2,g_3))+C_2(g_1,g_2g_3)\big). \end{aligned}$$

By (53), a defined by (61) is the associator iff

$$F_1(g_1)C_2(g_2,g_3)-C_2(g_1g_2,g_3)+C_2(g_1,g_2g_3)-C_2(g_1,g_2)+\partial C_3(g_1,g_2,g_3)=0.$$

This is exactly (55)–one of the conditions of the closedness of (C_2,C_3).

The naturality of the associator a is the following commutative diagram:

$$\begin{tikzcd} \big((g_1,\xi)\cdot_{\mathrm{h}}(g_2,\eta)\big)\cdot_{\mathrm{h}}(g_3,\gamma) \arrow[r, "a"] \arrow[d] & (g_1,\xi)\cdot_{\mathrm{h}}\big((g_2,\eta)\cdot_{\mathrm{h}}(g_3,\gamma)\big) \arrow[d] \\ \big((g_1,\xi+\partial m)\cdot_{\mathrm{h}}(g_2,\eta+\partial n)\big)\cdot_{\mathrm{h}}(g_3,\gamma+\partial k) \arrow[r, "a"] & (g_1,\xi+\partial m)\cdot_{\mathrm{h}}\big((g_2,\eta+\partial n)\cdot_{\mathrm{h}}(g_3,\gamma+\partial k)\big). \end{tikzcd}$$

To see that a is a natural isomorphism, we need to show that

$$a_{(g_1,\xi+\partial m),(g_2,\eta+\partial n),(g_3,\gamma+\partial k)}\cdot_{\mathrm{h}}\Big(\big((g_1,\xi,m)\cdot_{\mathrm{h}}(g_2,\eta,n)\big)\cdot_{\mathrm{h}}(g_3,\gamma,k)\Big) \tag{65}$$

is equal to

$$\Big((g_1,\xi,m)\cdot_{\mathrm{h}}\big((g_2,\eta,n)\cdot_{\mathrm{h}}(g_3,\gamma,k)\big)\Big)\cdot_{\mathrm{h}} a_{(g_1,\xi),(g_2,\eta),(g_3,\gamma)}. \tag{66}$$

By straightforward computations, we obtain that (65) is equal to

$$\big(g_1g_2g_3,\xi+F_1(g_1)(\eta)+F_1(g_1g_2)(\gamma),m+F_1(g_1)(n)+F_1(g_1g_2)(k)+F_2(g_1,g_2)(\gamma+\partial k)-C_3(g_1,g_2,g_3)\big),$$

and (66) is equal to

$$\big(g_1g_2g_3,\xi+F_1(g_1)(\eta)+F_1(g_1g_2)(\gamma),m+F_1(g_1)(n)+F_1(g_1)F_1(g_2)(k)+F_2(g_1,g_2)(\gamma)-C_3(g_1,g_2,g_3)\big).$$

Hence (65) is equal to (66) by (53).

By (53) and the fact that $F_1(1_x) = \mathrm{Id}$, we have

$$\begin{aligned}
(g,\xi)\cdot_{\mathrm{h}}\mathrm{inv}(g,\xi) &= (g,\xi)\cdot_{\mathrm{h}}\big(g^{-1}, -F_1(g^{-1})(\xi) - C_2(g^{-1},g)\big)\\
&= \big(gg^{-1}, \xi - F_1(g)F_1(g^{-1})(\xi) - F_1(g)C_2(g^{-1},g) + C_2(g,g^{-1})\big)\\
&= \big(1_{\beta(g)}, -\partial F_2(g,g^{-1})(\xi) - F_1(g)C_2(g^{-1},g) + C_2(g,g^{-1})\big)\\
&= \big(1_{\beta(g)}, -\partial F_2(g,g^{-1})(\xi) + \partial C_3(g,g^{-1},g)\big).
\end{aligned}$$

The last equality is due to (55) (notice that our cocycles are normalized). Thus the unit given by (64) is indeed a morphism from $(1_{\beta(g)}, 0)$ to $(g,\xi)\cdot_{\mathrm{h}}\mathrm{inv}(g,\xi)$.

To show the naturality of the unit, we need to prove

$$\big((g,\xi,m)\cdot_{\mathrm{h}}\mathrm{inv}(g,\xi,m)\big)\cdot_{\mathrm{v}} i_{(g,\xi)} = i_{(g,\xi+\partial m)},$$

i.e. the following commutative diagram:

$$\begin{tikzcd}
 & (1_{\beta(g)},0) \arrow[dl, "i_{(g,\xi+\partial m)}"'] \arrow[dr, "i_{(g,\xi)}"] & \\
(g,\xi+\partial m)\cdot_{\mathrm{h}}\mathrm{inv}(g,\xi+\partial m) & & (g,\xi)\cdot_{\mathrm{h}}\mathrm{inv}(g,\xi) \arrow[ll, "(g,\xi,m)\cdot_{\mathrm{h}}\mathrm{inv}(g,\xi,m)"']
\end{tikzcd}$$

This follows from

$$F_2(g,g^{-1})(\partial m) = F_1(g)\cdot F_1(g^{-1})(m) - F_1(g\cdot g^{-1})(m) = F_1(g)\cdot F_1(g^{-1})(m) - m,$$

which is a special case of (53).

Since $F(1_x) = \mathrm{Id}$ and our cocycle $C_2 + C_3$ is normalized, we have

$$(1_{\beta(g)},0)\cdot_{\mathrm{h}}(g,\xi) = (g,\xi), \quad (g,\xi)\cdot_{\mathrm{h}}(1_{\alpha(g)},0) = (g,\xi).$$

Hence the left unit and the right unit can also be taken as the identity isomorphisms.

The counit $e_{(g,\xi)} : \mathrm{inv}(g,\xi)\cdot_{\mathrm{h}}(g,\xi) \longrightarrow (1_{\alpha(g)},0)$ can be taken as the identity morphism since we have

$$\mathrm{inv}(g,\xi)\cdot_{\mathrm{h}}(g,\xi) = \big(g^{-1}, -F_1(g^{-1})(\xi) - C_2(g^{-1},g)\big)\cdot_{\mathrm{h}}(g,\xi) = (1_{\alpha(g)},0).$$

Lastly, we need to show

- the pentagon identity for the associator,
- the triangle identity for the left and right unit laws,
- the first zig-zag identity,
- the second zig-zag identity.

We only give the proof of the pentagon identity, the others can be proved in similar fashions and we leave them to the readers. The pentagon identity is equivalent to

$$\begin{aligned}
&a_{(g_1,\xi),(g_2,\eta),(g_3,\gamma)\cdot_{\mathrm{h}}(g_4,\theta)}\cdot_{\mathrm{v}} a_{(g_1,\xi)\cdot_{\mathrm{h}}(g_2,\eta),(g_3,\gamma),(g_4,\theta)} =\\
&\big((g_1,\xi)\cdot_{\mathrm{h}} a_{(g_2,\eta),(g_3,\gamma),(g_4,\theta)}\big)\cdot_{\mathrm{v}} a_{(g_1,\xi),(g_2,\eta)\cdot_{\mathrm{h}}(g_3,\gamma),(g_4,\theta)}\cdot_{\mathrm{v}}\big(a_{(g_1,\xi),(g_2,\eta),(g_3,\gamma)}\cdot_{\mathrm{h}}(g_4,\theta)\big)
\end{aligned}$$

The condition (55) implies that these elements can be vertical multiplied. Then by straightforward computations, the left hand side is equal to

$$\Big(g_1g_2g_3g_4, \xi + F_1(g_1)(\eta) + F_1(g_1g_2)(\gamma) + F_1(g_1g_2g_3)(\theta) + C_2(g_1,g_2) + C_2(g_1g_2,g_3) + C_2(g_1g_2g_3,g_4),$$
$$F_2(g_1g_2,g_3)(\theta) + F_2(g_1,g_2)\big(\gamma + F_1(g_3)(\theta) + C_2(g_3,g_4)\big) - C_3(g_1g_2,g_3,g_4) - C_3(g_1,g_2,g_3g_4)\Big),$$

and the right hand side is equal to

$$\Big(g_1g_2g_3g_4, \xi + F_1(g_1)(\eta) + F_1(g_1g_2)(\gamma) + F_1(g_1g_2g_3)(\theta) + C_2(g_1,g_2) + C_2(g_1g_2,g_3) + C_2(g_1g_2g_3,g_4),$$
$$F_2(g_1,g_2)(\gamma) + F_2(g_1,g_2g_3)(\theta) + F_1(g_1)\circ F_2(g_2,g_3)(\theta) - F_1(g_1)C_3(g_2,g_3,g_4)$$
$$-C_3(g_1,g_2g_3,g_4) - C_3(g_1,g_2,g_3)\Big).$$

By (54) and (56), they are equal.

Finally, it is not hard to verify that the natural inclusion $\iota_\bullet$ is injective and the natural projection of $\phi_\bullet$ is surjective. Moreover, $\ker(\phi_\bullet) = \mathrm{im}(\iota_\bullet) = E_\bullet$. Thus we indeed obtain an extension. ■

Remark 5.5 *We remark here how to go back from $\hat{G}$ to a Lie 2-algebroid. The systematical way to differentiate a Lie n-groupoid to an NQ manifold is described in [Ševa] in the language of graded manifolds. Here we only describe briefly the differentiation inspired by this work (using explicit words in usual differential geometry), and postpone the detailed calculation to future studies.*

Recall the differentiation of a Lie groupoid $t, s : H_1 \rightrightarrows H_0$ to a Lie algebroid. The Lie algebroid is $\ker Ts|_{H_0}$ containing tangent vectors of H_1 on H_0 vanishing along the source map. Similarly, we obtain a graded vector bundle

$$\ker T\alpha|_{G_0} \oplus \ker Ts|_{G_0}[1] = (A \oplus E_0) \oplus E_{-1}[1],$$

where A is the Lie algebroid of G. Now we explain how to obtain the Lie 2-algebroid structure (38) on this graded vector bundle. The anchor ρ of the Lie 2-algebroid is induced by the anchor of A.

We notice that the formula for the horizontal multiplication (59) is exactly the same as the formula for a usual extension of groupoid by a representation and a 2-cocycle. It implies the second formula of (38). Moreover, we notice that there is an (nonassociative) action of $G_1 \oplus E_0$ on E_{-1} given by the horizontal multiplication:

$$(g,\xi)\cdot m := pr_{E_{-1}}\big((g,\xi,0)\cdot_h(1_{s(g)},0,m)\big) = F_1(g)(m).$$

This implies the third formula of (38). The term l_3 is more difficult to explain, but at least in our case of Courant algebroids, it is determined by l_2 since $\partial : E_{-1} \to E_0$ is injective.

5.3 Application to Integration of Courant algebroids

Now we are ready to integrate the Courant algebroid $TM \oplus T^*M$ with Ševera class $[H]$. The integrating Lie 2-groupoid is simply the extension Lie 2-groupoid of $\Pi_1(M)$ by its representation up to homotopy $T^*M \xrightarrow{\mathrm{Id}} T^*M$ and the integrating 2-cocycle (C_2, C_3) of (c_2, c_3) coming from H. More explicitly, the Lie 2-groupoid is modeled on the action Lie

groupoid $(\Pi_1(M)\times_M T^*M)\rtimes_M T^*M \rightrightarrows \Pi_1(M)\times_M T^*M$ with T^*M acts on $\Pi_1(M)\times_M T^*M$ by addition on T^*M. The structure maps are given as in Prop. 5.4.

We now give a description in Kan complex using the correspondence in [Zhu09]. The 0-th level $X_0 = M$ is simply the base of the Courant algebroid. The first level is

$$X_1 = \Pi_1(M) \times_{t,M} T^*M(= \hat{G}_1),$$

with $d_1 = \beta$ and $d_0 = \alpha$ and s_0 the natural embedding $M \to \Pi_1(M) \times_{t,M} T^*M$. The second level is

$$X_2 = \big(\Pi_1(M) \times_{t,M,s} \Pi_1(M)\big) \times_{t\circ pr_l \times t\circ pr_r \times t\circ pr_l, M^{\times 3}} T^*M^{\times 3},$$

such that for a typical element $(\gamma_{01}, \gamma_{12}, \xi_{x_0}, \xi_{x_1}, m_{x_0}) \in X_2$,

$$\begin{aligned}
&d_0(\gamma_{01}, \gamma_{12}, \xi_{x_0}, \xi_{x_1}, m_{x_0}) = (\gamma_{12}, \xi_{x_1}),\\
&d_1(\gamma_{01}, \gamma_{12}, \xi_{x_0}, \xi_{x_1}, m_{x_0}) = (\gamma_{01}\gamma_{12}, \xi_{x_0} + C_2(\gamma_{01}, \gamma_{12}) + F_1(\gamma_{01})(\xi_{x_1}) + id(m_{x_0})),\\
&d_2(\gamma_{01}, \gamma_{12}, \xi_{x_0}, \xi_{x_1}, m_{x_0}) = (\gamma_{01}, \xi_{x_0}).
\end{aligned}$$

Then in general X is determined by the first three levels,

$$X = cosk_3 sk_3(\Lambda[3,0](X), X_2, X_1, X_0).$$

That is X_n is a fibre product made up by X_2's, X_1's and X_0's. In fact, in this case, since the differential from T^*M to T^*M is an isomorphism, X_2 is totally determined by its images under d_0, d_1 and d_2. More explicitly there is a simplicial manifold $Y_\bullet$ with

$$\begin{aligned}
Y_n =&\Pi_1(M)^{\times n} \times_{M^{\times \binom{n+1}{2}}} T^*M^{\times \binom{n+1}{2}}\\
=&\{(\gamma_{0,1}, \gamma_{1,2}, \dots, \gamma_{n-1,n}; \dots \xi^{i,j}, \dots) : 0 \le i < j \le n, \gamma_{i,i+1} \in \Pi_1(M) \text{ is represented by}\\
&\text{a path from } x_i \text{ to } x_{i+1}, \text{ and } \xi^{i,j} \in T^*_{x_i}M.\}
\end{aligned}$$

One should imagine each element as the dimensional-1-skeleton of a n-polygon in M with each edge attached with a cotangent vector at the end. The face and degeneracy maps are naturally given by

$$d_k(\gamma_{0,1}, \gamma_{1,2}, \dots, \gamma_{n-1,n}; \dots \xi^{i,j}, \dots) = (\dots, \gamma_{k-1,k} \cdot \gamma_{k,k+1}, \dots; \dots, \hat{\xi}^{i,k}, \dots, \hat{\xi}^{k,j}, \dots),$$

$$s_k(\gamma_{0,1}, \gamma_{1,2}, \dots, \gamma_{n-1,n}; \dots \xi^{i,j}, \dots) = (\dots, \gamma_{k-1,k}, 1_{x_k}, \gamma_{k,k+1}, \dots; \dots, \tilde{\xi}^{i,j}, \dots),$$

with $\tilde{\xi}^{i,j} = \xi^{i,j}$ for $i < j \le k$, $\tilde{\xi}^{k,k+1} = 0$, $\tilde{\xi}^{i,j} = \xi^{i-1,j-1}$ for $k < i < j$, $\tilde{\xi}^{i,j} = \xi^{i,j-1}$ for $i \le k \le j-1$. Since $Y_\bullet$ is determined by its 1-skeleton, it is clearly a Lie 2-groupoid. Moreover, regardless of the cocycle (C_2, C_3), we have $X_\bullet \cong Y_\bullet$ as a simplicial manifolds since both are determined by their 1-skeleton and $X_2 \cong Y_2$. If we take a local neighborhood of $Y_0 = M$ in Y_n, we arrive at the local Lie 2-groupoid $\mathcal{TM}$ in [LBŠ], which differentiates to the standard Courant algebroid $(T^*[2]T[1]M, [H])$. Thus we have

Theorem 5.6 *A standard Courant algebroid $(T^*[2]T[1]M, [H])$ integrates to the semidirect product Lie 2-groupoid of $\Pi_1(M)$ with its coadjoint representation up to homotopy $T^*M \xrightarrow{\mathrm{Id}} T^*M$, regardless of the Ševera class $[H]$.*

Remark 5.7 (Comparation with other works) *The authors in [LBŠ] also construct a global version using the pair groupoid rather than the fundamental groupoid as we do in our approach. But there is no fundamental difference. This global Lie* 2*-groupoid built upon the pair groupoid is also the integration object of Mehta and Tang [MT, (1.3)]. We choose the fundamental groupoid in hope that we would achieve a more universal object, possibly with a universal symplectic structure. However, it seems that we need to go further (to "fundamental* 2*-groupoid") to achieve this universal object, since our current object is not source* 2*-connected. We postpone the investigation of this question to future works.*

References

[ACa] Camilo Abad and Marius Crainic. Representations up to homotopy and Bott's spectral sequence for Lie groupoids. arXiv:0911.2859.

[ACb] Camilo Abad and Marius Crainic. Representations up to homotopy of Lie algebroids. arXiv:0901.0319., to appear in *J. Reine. Angew. Math.*

[ASa] Camilo Arias Abad and Florian Schaetz. Deformations of Lie brackets and representations up to homotopy. arXiv:1006.1550.

[ASb] Camilo Arias Abad and Florian Schaetz. The A_∞ de Rham theorem and integration of representations up to homotopy. arXiv:1011.4693.

[BL04] John C. Baez and Aaron D. Lauda. Higher-dimensional algebra. V. 2-groups. *Theory Appl. Categ.*, 12:423–491 (electronic), 2004.

[GSM10] Alfonso Gracia-Saz and Rajan Amit Mehta. Lie algebroid structures on double vector bundles and representation theory of Lie algebroids. *Adv. Math.*, 223(4):1236–1275, 2010.

[Gua] Marco Gualtieri. Generalized complex geometry. arxiv:math.DG/0401221.

[Hit03] Nigel Hitchin. Generalized Calabi-Yau manifolds. *Q. J. Math.*, 54(3):281–308, 2003.

[KW01] Michael K. Kinyon and Alan Weinstein. Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces. *Amer. J. Math.*, 123(3):525–550, 2001.

[LBŠ] David Li-Bland and Pavol Ševera. Integration of Exact Courant Algebroids. arXiv:1101.3996.

[LM95] Tom Lada and Martin Markl. Strongly homotopy Lie algebras. *Comm. Algebra*, 23(6):2147–2161, 1995.

[LWX97] Zhang-Ju Liu, Alan Weinstein, and Ping Xu. Manin triples for Lie bialgebroids. *J. Diff. Geom.*, 45(3):547–574, 1997.

[Mac05] Kirill C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[MT] Rajan Amit Mehta and Xiang Tang. From double Lie groupoids to local Lie 2-groupoids. arXiv:1012.4103.

[Roy] Dmitry Roytenberg. Courant algebroids, derived brackets and even symplectic supermanifolds. PhD thesis, UC Berkeley, 1999, arXiv:math.DG/9910078.

[Roy02] Dmitry Roytenberg. On the structure of graded symplectic supermanifolds and Courant algebroids. In *Quantization, Poisson brackets and beyond (Manchester, 2001)*, volume 315 of *Contemp. Math.*, pages 169–185. Amer. Math. Soc., Providence, RI, 2002.

[RW98] Dmitry Roytenberg and Alan Weinstein. Courant algebroids and strongly homotopy Lie algebras. *Lett. Math. Phys.*, 46(1):81–93, 1998.

[Ševa] Pavol Ševera. L-infinity algebras as 1-jets of simplicial manifolds (and a bit beyond). arXiv:math/0612349 [math.DG].

[Ševb] Pavol Ševera. Letter to Alan Weinstein. http://sophia.dtp.fmph.uniba.sk/˜severa/letters/no8.ps.

[Šev05] Pavol Ševera. Some title containing the words "homotopy" and "symplectic", e.g. this one. In *Travaux mathématiques. Fasc. XVI*, Trav. Math., XVI, pages 121–137. Univ. Luxemb., Luxembourg, 2005.

[SZa] Yunhe Sheng and Chenchang Zhu. Integration of semidirect product Lie 2-algebras. arXiv:1003.1348.

[SZb] Yunhe Sheng and Chenchang Zhu. Semidirect products of representations up to homotopy. *Pacific J. Math.*, 249 (1), 211-236, 2011.

[Vor] Theodore Th. Voronov. Q-manifolds and Higher Analogs of Lie Algebroids. XXIX Workshop on Geometric Methods in Physics. AIP CP 1307, pp. 191-202, Amer. Inst. Phys., Melville, NY, 2010.

[WZ] Christoph Wockel and Chenchang Zhu. Integrating central extensions of Lie algebras via group stacks. work in progress.

[Zhu09] Chenchang Zhu. n-groupoids and stacky groupoids. *Int. Math. Res. Not. IMRN*, 2009(21):4087–4141, 2009.